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A
TREATISE
ON
ALGEBRA,
IN PRACTICE AND THEORY,

IN TWO VOLUMES,

WITH

NOTES AND ILLUSTRATIONS;

CONTAINING

A VARIETY OF PARTICULARS RELATING TO THE DISCOVERIES AND
IMPROVEMENTS THAT HAVE BEEN MADE IN THIS
BRANCH OF ANALYSIS.

BY JOHN BONNYCASTLE,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH.

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HIS ROYAL HIGHNESS

THE PRINCE OF WALES,

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INTRODUCTION.

THE earliest treatise on Algebra, which has come down to the present times, is that of Diophantus of Alexandria, in Egypt, who flourished about the middle of the fourth century after Christ, and wrote a work on this subject, in the Greek language, which, according to the testimony given by himself, in his introductory address to Dionysius, consisted, originally, of thirteen books, though, unfortunately for the interests of science, only the first six, and an imperfect tract on triangular and polygonal numbers, are now extant; which were first printed, in a Latin translation, by Xylander, in 1575, and afterwards, in Greek and Latin, with a comment, in 1621 and 1670, by Bachet and Fermat (*a*).

Other works on this subject, however, of a more easy and elementary kind, must, evidently, have existed long before the time of Diophantus; since he no where treats, in the abovementioned per-

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(*a*) This writer had a number of interpreters among the ancients, whose works are now lost: of which we have more particularly to regret the commentary of the learned Hypatia, the daughter of Theon the philosopher (anno Domini 410); whose talents, virtues and misfortunes, will ever entitle her to the homage of posterity. — See, for a more particular account of this illustrious victim of fanaticism, Bossut's History of Mathematics, English edition, 8vo. 1803.

formance, of the first principles and leading rules of the science, as a writer in the infancy of the art would have done, but proceeds, at once, to the resolution of a particular class of indeterminate problems, relating chiefly to the finding of square and cube numbers, which, even at present, are generally considered as forming one of the most difficult branches of pure analysis.

But whether we are indebted for this admirable invention to the genius of the Greeks, as has, hitherto, been commonly thought, or to that of some other ancient nation, cannot, at this distance of time, be easily ascertained; though, from the information which, for more than a century past, has been gradually obtained through our intercourse with the East, there are strong reasons for believing that algebra, as well as our common system of arithmetic, originated among the Hindoos, or natives of India; who are known to possess some very valuable works on this subject, which contain rules and principles that do not appear to have been derived from any foreign source.

The chief of these, and indeed the only performances of the kind, that have come to our knowledge, are the two celebrated works called the *Beja Ganita*, and the *Lilavati*; the first of which treats wholly on algebra and its applications, and the latter on arithmetic, algebra and mensuration, or practical geometry; and were both of them written in Sanscrit, about the end of the twelfth, or beginning of the thirteenth century of the Christian era, by Bhascara Acharya, a famous Hindoo astronomer and mathematician of the city of Biddur, in the country of the Deccan.

It is chiefly, however, from the Persian translation of the Beja Ganita, which was made in 1634, by Ata Allah Rusheedee, at Agra or Delhi, and that of the Lilavati, by the celebrated Fyzee, in 1587, that we have obtained any certain knowledge of the contents of these curious performances; for which we are principally indebted to Mr. Edward Strachey, of the East India Company's Bengal civil establishment, who has lately favoured the public with a very interesting account of the former of these works, and some parts of the latter, taken from the source above mentioned (*b*).

From this publication it appears, that besides a knowledge of most of the common rules of algebra, at present in use, exclusively of cubic and biquadratic equations, the Hindoos were acquainted with some branches of the science that were not known in Europe till about the middle of the eighteenth century; it being here shown, by a va-

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(*b*) The late Mr. Reuben Burrow, who resided several years in India, collected, in that country, a number of Oriental manuscripts on mathematical subjects; some of which he left to his friend Mr. Dalby, Professor of Mathematics in the Royal Military College, Wycombe; particularly fair copies, in Persian, of the Beja Ganita and Lilavati, with the English of each word, written above the Persian, which Mr. Strachey mentions as having been of some service to him in composing his present performance.

riety of examples, extracted from the Persian translation of the Beja Ganita, that they were able to resolve all the possible forms of indeterminate equations, both of the first and second degree, by methods that were successively invented by Bachet de Mezeriac, Fermat, Euler and Lagrange, at a much later period; and as the questions here given, as well as the manner of treating them, are different from those of Diophantus, it is scarcely to be doubted that these people had among them a number of propositions relating to this science, which were of their own invention.

The Beja Ganita, it is true, as well as the Lilavati, are works of comparatively modern date; but from Bhascara's own account, who assumes no other character than that of a compiler, it appears that they were extracted from other performances of the same kind, which existed in his time; so that it is highly probable there are much older Indian treatises on algebra, in that country, than are yet known to us (c); and it is to be hoped, as the importance of this inquiry is more generally felt, that we shall, ere long, obtain entire translations of the originals, or such accounts of them as may enable us to determine, with more certainty, to whom we are indebted for a science, which, in its present state, may be justly considered as doing the highest honour to the reasoning and inventive powers of man.

(c) Mr. Davis, the well known author of two papers on Indian astronomy, in the third volume of the Asiatic Researches, who has furnished Mr. Strachey with a number of extracts and translations from the original Sanscrit Beja Ganita, which are given by him at the end of his present work, observes, "that almost any trouble and expense would be compensated by the possession of the three copious treatises on algebra from which Bhascara declares he extracted his Beja Ganita; and which, in this part of India, are supposed to be entirely lost."

This information is the more desirable, as it is acknowledged by Mr. Strachey, that the Persian translations of these works, though highly valuable in many respects, do not convey so correct an idea of the original performances as could be wished; as they chiefly consist of an undistinguished mixture of text and commentary, and, in some places, even refer to Euclid; though Mr. Davis, who made the extracts from the original Beja Ganita, mentioned in the last note, is decidedly of opinion that the main part of the Persian translation is taken from the Sanscrit work, and has no doubt that the science it contains is of Hindoo origin; the references above spoken of, and other discrepancies, being considered by him as interpolations of the translator.

But, not to dwell upon this part of the subject, which, it must be confessed, is still attended with some obscurity, it is well known, that, in whatever age or country algebra was invented, both the name and the science were first made known to us, about the end of the eleventh century, by the Arabians, or Moors, who were then settled in Spain; to which country a number of eminent men, of that time, repaired, in order to obtain, from these people, a knowledge of the several branches of literature and science which they had brought with them from the East; and among the rest of algebra, which soon afterwards began to be cultivated in most parts of Europe with great ardour and success (*d*).

Italy, however, appears to have taken the lead in this respect; the art, according to the testimony of Signor Cossali (History of Algebra, 2 vols. 4to., 1797) having been first introduced into that country, as early as in the beginning of the thirteenth century, by Leonard Bonacci, commonly called Leonard of Pisa, an Italian merchant, who, by visiting the sea-ports of Africa and the Levant, acquired a knowledge of such branches of the mathematical sciences as were then known; which he afterwards disseminated among his countrymen, by

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(*d*) The etymology of the name algebra has been variously given by different authors; but it appears evident, from Mr. Strachey's account, that the word is purely Arabic, being compounded of the article *al*, and *jebr*, *restoration*; which denotes one of the modes of reducing equations, by transposing or adding the negative terms, so as to make them all affirmative. The same writer is also of opinion, that, although we received this science from the Arabs, neither that people nor the Persians can be considered as the inventors, as their knowledge of the subject appears to have been of a very limited kind, compared with that of the Hindoos.

composing, in the year 1208, a work on arithmetic, and in 1228 another on arithmetic and algebra; which performances the author above quoted considers as very regular and orderly treatises, for the time in which they were written.

Many other manuscript writings, of a similar nature, also appeared in Tuscany and other parts of Italy, soon after the time above mentioned; but the first printed works on this subject, are those of Lucas Paciolus, or Lucas de Burgo, a Cordelier or Minorite friar, commonly called Luca del Borgo di san Sepolcro; who published several treatises on arithmetic, algebra and geometry, in the years 1470, 1476, and 1481; but his principal work is that entitled *Summa de Arithmetica, Geometria, proportioni et proportionalita*, printed in folio, at Venice, 1494.

From this performance, which exhibits the state of the science at the beginning of the sixteenth century, it appears that the earliest European analysts, as well as the Arabs, from whom they derived the greater part of their knowledge, used no symbols or signs for either quantities, or the operations that it was necessary to perform on them, excepting a few contractions for the words or names themselves; and that the art, at that time, extended no farther than to the resolution of numeral equations of the first and second degree; in the latter of which, they used the double values of the unknown quantity, in the case where they are both positive; but take no notice of such of the roots as are negative or imaginary (e).

The passage, indeed, to the higher orders of equations, was a matter of considerable difficulty, which, in all probability, had never hitherto been made; but this was in part effected, soon after the publication of the work last mentioned, by Scipio Ferreus, professor of mathematics at Bologna, in Italy, who first discovered, about the year 1505, a rule for resolving one of the cases of a compound cubic equation; and about thirty years afterwards, one of his disciples, of the name of Florido, to whom he had shown his method, having proposed, to Nicholas Tartalea, a celebrated mathematician of Brescia, by way of challenge, several questions, the solutions of which depended upon this formula, Tartalea not only discovered the rule for resolving them, but also those for the other cases; which enabled him, in his turn, to confound his opponent, by proposing to him some questions of a different kind, which the latter could not answer.

But this, as well as several other branches of mathematical knowledge, was more particularly cultivated and improved, soon after the time above mentioned, by Hieronymus Cardan, of Bononia, an eminent Italian writer, of great scientific acquirements, who first published, in the year 1539, a work on arithmetic and algebra, in nine books, in the Latin language, at Milan, where he practised physic and read mathematical lectures; and in a new edition of the same performance, printed in 1545, he gave a tenth book, containing the whole doctrine of cubic equations, which had been chiefly revealed to him by Tartalea, under an oath of secrecy, about the time of the publication of his first nine books (*f*).

(e) The most remarkable circumstance in the algebra of this author is, that in the denoting of powers, he compounds the names or indices according to the multiplication of the numbers 2, 3, 4, &c. and not by their addition, as was done by Diophantus; which induced Dr. Wallis to adopt the opinion that the knowledge we first obtained of this science was not derived from the Greeks.

In this work, which is a very regular and masterly performance, Cardan lays down rules for the solution of all the forms and varieties of cubic equations, whether complete or incomplete, which he demonstrates geometrically: he also treats of nearly all the transformations and methods of managing equations that are known to us at present; showing, among other things, that he was acquainted with the number and nature of their

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(*f*) For a farther account of this transaction, which was highly resented by Tartalea, see Montucla's *Histoire des Mathématiques*, vol. i, p. 591, et seq. second edition; also Hutton's Mathematical Dictionary, Art. Algebra; where a full detail is given of all the most remarkable circumstances relating to this curious business, as well as a copious analysis of Cardan's and other performances of that time; to which article I am indebted for several particulars respecting the improvements and discoveries of some of the earlier European writers on this subject.

roots, both positive and negative, and knew that they fail or become imaginary in pairs; as also that the even roots of positive quantities are either affirmative or negative; that the odd roots of negative quantities are real and negative; and that the even roots of them are imaginary. It also appears that he had a rule for determining the cube roots of such binomials as admit of extraction; and that he frequently used the literal notation a, b, c, d, &c. He likewise applied algebra to the resolution of geometrical problems; and was well acquainted with the difficulty of what is usually called the *irreducible case*, in cubic equations, upon which he spent a great deal of time, in endeavouring to obtain its solution.

To this we may also farther add, that the resolution of biquadratic equations, or those of the fourth order, followed closely that of equations of the third; it being to Lewis Ferrari, a young man of great talents, who was one of the disciples of Cardan, that we are indebted for this discovery; the particulars of which Cardan has detailed in the thirty-ninth chapter of his Algebra, where he has demonstrated the practical rule given by Ferrari for this purpose, as well as illustrated it by a number of numeral examples, in all the several cases of those equations.

After this last publication, the next printed work on the subject, in the order of time, is by Tartalea, who, as before observed, was contemporary with, though probably something older than Cardan; but we do not know of any treatise of his on algebra before that which appeared in 1546, the year after the date of Cardan's performance above mentioned; when he published his *Quesiti è inventioni diversæ*, at Venice, where he resided as a public lecturer on mathematics; in which work it is observable, that Tartalea made no alteration in the notation, or forms of expression, used by Lucas de Burgo; but generally writes the names of all the operations in words at length, without using any contractions, except the letter ℞ for root, or radicality; so that the only remarkable things in this treatise are the discovery of the rules for the solution of cubic equations, and the curious circumstances attending the correspondence and conferences which took place between him and Cardan, on that account.

About the same time also, that Cardan and Tartalea flourished in Italy, the science of algebra began to be much cultivated and improved in Germany; particularly by Stifel and Scheubel; the former of whom published an excellent treatise on this subject entitled *Arithmetica Integra*, printed at Noremberg, in 1544, being a year before the publication of Cardan's last work; from which it appears that he was the first who employed the characters +, −, √, for plus, minus, and root; as also the numeral indices, or, as he first calls them, exponents, −3, −2, −1, 0, 1, 2, 3, &c. for powers, as far as regards integral numbers, but not fractional ones. He likewise used the literal notation A, B, C, &c. for different unknown quantities; but neither this writer nor Scheubel, who published a work on the same subject, in 1552, speak of cubic equations.

A few years after the appearance of these treatises in Italy and Germany, Robert Recorde, a native of Wales, who styles himself a teacher of mathematics and practitioner in physic at Cambridge, proved by his writings, that algebra was not altogether unknown in England at that time; as appears by the second part of his arithmetic, which was published in the year 1557, under the quaint title of "The Whetstone of Witte, containing the Extraction of Rootes, the Cossike Practice, with the Rule of Equation, and the Workes of Surde Nombers;" in which performance he treats the principal part of his subject in nearly the same manner as Stifel and Scheubel, whom he frequently quotes; but the chief things which are new in this author, are the extraction of the roots of compound algebraical quantities, the use of the terms binomial and residual, and the introduction of the sign = for equality.

There also appeared, in 1558, a work on this subject by James Pelletier, commonly called Peletarius, of Mans, in France, who treats, in a very able and methodical manner, of most parts of the science then known, except cubic equations; which he wholly omits to mention, though Cardan's work had been published several years before. In other respects, however, this may be considered as a neat and well digested treatise; containing several useful improvements; he having here first shown that any rational root of an equation is one of the divisors of its last term; besides which he teaches how to reduce binomial and trinomial surds to their most simple terms, by multiplying them by compound factors, and has pointed out some curious properties of square and cube numbers, showing, among other things, how to form any series of them, by adding together the several orders of their differences (*g*).

Peter Nunez, or, as he is more frequently called, Nonius, professor of mathematics in the university of Coimbra, in Portugal, likewise published, at Lisbon, in 1564, a work on algebra, in the Castilian language; in which he chiefly follows the manner of De Burgo, Cardan, and Tartalea; making a number of critical remarks on many parts of their performances, but without adding any discoveries or remarkable improvements of his own; nor does he mention any preceding writer on this subject, either among the natives of Spain or the Moors; which is the more singular, as the latter of these people occupied a great part of the Peninsula for several centuries, and are supposed to have first brought the science from the East into that country;

(*g*) We might here also have mentioned Peter Ramus, another French writer of some note about this time; who published a treatise on arithmetic and algebra in the year 1560; but this work is merely elementary, and contains nothing particularly worthy of notice.

where it might have been supposed some remains of it would be found.

The earliest European writers, indeed, on this subject, appear to have been those of Italy; among whom we may here farther reckon Raphael Bombelli, of Bologna, who, in his Algebra, printed in 1572, first demonstrated, from geometrical principles, that the terms of the formula, which represents one of the roots in the irreducible case of cubic equations, form, when taken together, a real result; in support of which proof he lays down a rule for the extraction of the roots of certain binomial surds, and shows that the radical parts, in some particular numeral examples of the kind above mentioned, will destroy each other. He here likewise first observes, that angles may be trisected by means of a cubic equation; but excepting these particulars, and his method of denoting the unknown quantity and its powers, by the character ⌣, with the figures 1, 2, 3, &c. set over it, this work differs but little from those of preceding writers.

A very ingenious treatise on arithmetic and algebra was also published, in the Flemish dialect, in 1605, by Simon Stevin, of Bruges; in which, among other things of a curious and original nature, he has greatly improved the notation of powers, first given by Stifel for integral exponents, by placing the figures that represent them, in a small circle, which is made to denote the unknown quantity, and extending them to fractional indices, so as to denote all kinds of roots in the same way as powers; an improvement which has commonly been thought of much later date: besides which, he improved and extended the use and notation of coefficients, including in them fractions, radicals, and all sorts of numbers in general. He here likewise teaches the generation of powers, by means of their coefficients, placed in the form of a triangle, and makes use of the same numbers for extracting roots; both of which things, however, had been before done by Stifel (*h*).

But it is chiefly to the celebrated Vieta, whose algebraical works were written about the year 1600 (*i*), though not printed till after his death, which happened in 1603, that we are indebted for having first generalized the algorithm of this science, by denoting all kinds of quantities, whether known or unknown, by the capital letters of the alphabet; using the consonants B, C, D, &c. for given quantities, and the vowels A, E, I, &c. for those that are unknown; which improvement, by its near approach to our present method of notation, may be considered as one of the greatest steps that had

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(*h*) A complete edition, in French, of the works of this writer, was printed at Leyden, in 1634, with notes and additions, by Albert Girard; in which is to be found a translation of all the six books of Diophantus.

(*i*) The whole mathematical works of Vieta were collected together by Francis Schooten, and elegantly printed at Leyden, in 1646, in a folio volume.

yet been made in analysis. Besides this, he also showed how the roots of equations may be augmented or diminished in any given proportion, without knowing the roots themselves; and laid down a method of deriving the higher orders of equations from quadratics, in a more scientific way than that used by Harriot: he likewise determined the roots of compound equations, by a method of approximation similar to that for pure powers, and, in some particular cases, constructed them by means of angular sections, as before adverted to by Bombelli. Lastly, he employed several terms and expressions, in treating of this subject, which are still in use; such as coefficient, affirmative and negative, pure and affected, &c. and was the first who used a line, or vinculum, for connecting compound quantities.

Next after Vieta may be reckoned Albert Girard, an ingenious mathematician of the Low Countries, who, in his work entitled *Invention Nouvelle en l'Algebre*, printed at Amsterdam, in 1629, introduced several new discoveries and improvements into this science; among which, it may be observed, that he appears to have been the first who understood the general doctrine of the formation of the coefficients of the terms of an equation from the sums of their roots, products, &c.; as also, that every equation has as many roots, real and imaginary, and no more, as there are units in the index of its highest power. Besides which, he has here also resolved, for the first time that we know of, the irreducible, and all the other cases of cubic equations, by means of a table of sines; and was likewise the first who discovered a method for determining the sums of the powers of the roots of any equation in terms of the coefficients.

Among the most distinguished analysts, however, of this period, we may reckon our countryman the celebrated Harriot, who, in his *Artis Analyticæ Praxis*, published by his friend Walter Warner, in 1631 (*k*), first introduced the uniform use of the small letters of the alphabet *a*, *b*, *c*, *d*, &c.; using the vowels *a*, *e*, *i*, &c. for unknown quantities, and the consonants *b*, *c*, *d*, &c. for the known ones, which he joined together, like the letters of a word, to represent their products, prefixing the coefficients as we do at present, except by separating it from the rest by a point. He also first employed the characters > and < for greater and less; and in the reduction of equations, he placed all their terms on one side, and arranged the operations in separate steps, or lines, setting the explanations in the margin, on the left hand, for each line; by which, and other advances, he may be considered as having given the science that form which it has ever since

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(*k*) This work did not make its appearance till ten years after the author's death, which was six years after the first publication of the principal parts of Vieta's Algebra, by Alexander Anderson; so that as Harriot's performance must have been written a considerable time before this, it is probable that he had never seen what the former had done upon this subject.

retained. He likewise farther showed how all the higher orders of equations may be produced by the continued multiplication of those of the first order; so as to exhibit to the eye, by the bare inspection of the operation, the nature and number of their roots, with the composition and relations of the coefficients of the several terms; from which, though not so strict a method of investigation as the subject requires, many of the most important properties of equations have since been derived. He also greatly improved the extraction of the roots of numeral equations of different orders, by clear and explicit rules, derived from the mode in which the equations are generated, and illustrated most of the principles he employed by a number of practical examples.

William Oughtred, another English mathematician, of considerable talents, who was contemporary with Harriot, also published in the same year, 1631, a work on this subject, entitled *Clavis Geometrica*, in which he chiefly follows Vieta, in his notation by capitals, though with some variations. Among other improvements, he appears to have been the first who set down decimal fractions without their denominators, by separating them from integers by a bracket instead of a point, as 21(56. In algebraic multiplication, also, he either joins the letters together, which represent the factors, like a word, or connects them by the mark ×; which is the earliest use we find of this character. He likewise appears to have been the first who used points to denote proportion, as in the case 7 . 9 : : 28 . 36; and for continued proportion he employs the mark ∺: he also denoted roots by the common radical sign √, with *q* placed after it for square root, *c* for cube root, *qq* for biquadrate, &c.; it is here, likewise, that we meet with the first instance of applying algebra to geometry, in the investigation of new geometrical properties.

About the same time, several other writers also published some very useful treatises for diffusing a knowledge of this and other branches of science; among whom we are more particularly indebted to Herigone, who printed at Paris, in 1634, in five volumes 8vo, the first course of mathematics that had hitherto appeared; which performance, both in form and substance, may be considered as a very ingenious work, for that period; containing, among a variety of other subjects, a short treatise on algebra; in which he employs the small letters of the alphabet for designating both known and unknown quantities, as had been done in the Algebra of Harriot, published only three years before: he has also introduced into this part of his work several convenient characters and abbreviations of his own, for abridging the operations; and in the notation of powers, he does not repeat the letters, like Harriot, but denotes them by their numeral exponents, as Descartes did three years afterwards.

But no work, perhaps, has contributed more to the general advancement of this science, in all its several branches, than the Geometry of Descartes, first published in 1637; in which performance, besides the application of algebra, here first made, to the doctrine of curves, we find a number of improvements in the rules and methods that had been before laid down by Cardan, Vieta, Harriot and others, for the transformation and management of equations; which he generally treats in a more clear and explicit manner than had usually been done: he here, also, more distinctly particularizes the negative and imaginary roots of equations, which had been hitherto but slightly attended to; and shows that they are all as real and proper for resolving questions as positive roots; the distinction that ought to be made between them having no other foundation than the different manner of considering the quantities of which they are the symbols. Besides which, he taught how to distinguish, in an equation containing only real roots, the number of positive roots, and that of the negative ones, by the permanencies or changes in the signs of its terms; and was the first who fully developed the method of indeterminate coefficients, by applying it to the resolution of biquadratic equations, which he effected by means of a cubic and two quadratics (*l*).

(*l*) Descartes, among his other improvements, has commonly been represented, by most foreign writers on this subject, as having been the first who introduced the general method of denoting the powers of letters by their numeral exponents; but this, as was shown in the last article, had been before done by Herigone.

To this we may farther add, that, soon after the publication of the treatise of Descartes above mentioned, a number of other writers, following the course he had pointed out, endeavoured, by applying themselves to the new geometry, to render both this and the algebraical branch of it more perfect and consonant with each other; among whom may be reckoned Francis Schooten, Professor of Mathematics in the university of Leyden, who first translated this celebrated work from French into Latin, in 1649, with a commentary of his own and some notes of M. de Beaune; and, in 1659, he gave a new edition of it, in two volumes, with the addition of some posthumous tracts of the last mentioned writer, on the nature and constitution of equations; showing how to assign the limits of their greatest and least roots, and other particulars.

He likewise inserted, in the same work, two very interesting letters of Hudde, a burgomaster of Amsterdam, which contain, among other things relating to the drawing of tangents, and the maxima and minima of variable quantities, a beautiful theorem for resolving equations that have equal roots; showing that if each term of such an equation be multiplied by the exponent of the unknown quantity in that term, the resulting equation may be depressed one degree lower than the proposed equation (*m*). Besides which he also annexed

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(*m*) It was also farther shown by Rolle, in his Algebra, printed in 1690, that the roots of the equation, thus formed, to it a valuable tract of Van Heuraet's on the rectification of curve lines, and another on the elements of curves, by the celebrated Dutch pensionary De Witte; adding, at the same time, several useful notes and observations of his own.

We have here likewise to mention the celebrated Fermat, counsellor of the parliament of Toulouse, in France, who was contemporary with Descartes, and a competitor with him for some of his brightest discoveries, which he was in possession of, though he had not published them, before the Geometry of the latter had appeared; namely, the application of algebra to curve lines, which he expressed by algebraical equations, and thence developed the properties of various kinds of plane and solid figures, which were, at that time, new to the science: he had also an ingenious method of drawing tangents to curves, as well as another for resolving problems of the kind usually called *de Maximis et Minimis*, each of which bears a great resemblance to the differentiel calculus, or method of fluxions, both in the manner of treating these problems, and in the algebraical notation and process (*n*).

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are the limits of the roots of the proposed equation; so that the determination of these limits depends upon the solution of an equation of the next inferior degree; and the limits of this upon one a degree lower, and so on.

(*n*) The works of Fermat were printed after his death, at Paris, in two volumes folio, 1665; the first of which consists of a new edition of Diophantus, enriched with his notes and discoveries; and the other contains his own proper writings on

The impulse thus given to this branch of the science, was still farther promoted by the publication, in 1655, of the *Arithmetica Infinitorum* of Wallis; which was a new method of reasoning on quantities, of a more clear and satisfactory kind than that used by Cavalerius, in his *Geometria Indivisibilibus*, 4to, Bononia, 1635, and which may be considered as having led the way to most of the great improvements that have since been made in the higher branches of analysis; particularly in the doctrine of series, of which he was, in a great measure, the inventor, at least as far as regards the higher and more difficult kinds. Besides which it may be observed, that he has here given, for the first time, an expression for exhibiting the area of the circle in an infinite series, as well as a variety of other problems relating to quadratures and rectifications in general. He also substituted fractional indices in the place of the radical signs, by which the algorithm and practical operations of the science are greatly facilitated and abridged (o).

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geometry and the modern analysis, together with the very interesting correspondence that passed between him and Marsenne, Pascal, Roberval and others, respecting the improvements and inventions which each of them professed to claim.

(o) Besides this and various other performances on mathematical subjects, which were all printed together at the university press, Oxford, in three volumes folio, 1699, Wallis also published, in 1685, his History and Practice of Algebra, in one volume folio; which is a work full of learned and useful information, but deficient in point of research, as far as regards the discoveries and improvements of some of the earlier writers on this subject.

About this time, also, Demoivre gave various papers in the Philosophical Transactions, containing a number of useful improvements and discoveries in algebra; *viz.* in 1697, a method of raising a multinomial, or infinite series, to any power, or extracting any root of it; in 1698, the extraction of the root of an infinite equation; in 1707, analytical solutions of certain equations of the 3d, 5th, 7th and other odd dimensions; in 1722, a piece on algebraic fractions and the summation of recurring series; and, in 1738, the reduction of radicals into more simple forms. He also published, in 1730, an entire treatise on these and other similar subjects, entitled *Miscellanea analytica de Seriebus et Quadraturis*, which is a work of great merit, abounding with many new inquiries and investigations of the most interesting kind.

In 1707, Whiston also published Newton's *Arithmetica Universalis*, which work was the text book used by our author, at the time he was Professor of Mathematics in the university of Cambridge; and although it was never intended to be made public, it abounds with many great and useful improvements; particularly in the transformations of equations, the limits of their roots, the number of impossible roots, and the use of rational and surd divisors. It also contains a select collection of algebraical and geometrical problems, which are resolved in the most elegant and masterly manner; as likewise the linear construction of equations of different orders, and a general method of determining the roots of numeral equations by approximation; which last subject was also fully treated of by Ralphson, in his *Analysis Equationum universalis*, printed in 1690.

But of all the discoveries of Newton in this science, none have contributed so much to the general advancement and perfection of analysis, as that of his celebrated formula, known under the name of the Binomial Theorem; which, besides its more immediate use in determining any power or root of a compound quantity, has also been found of the greatest service in many inquiries of a higher and more difficult kind, where scarcely any other means of investigation could have been employed. We are likewise farther indebted to him for another important discovery in this branch of the subject, called the Reversion of series, or the method of finding the value of the root of an infinite series, in terms of another series of a different kind; both of which inventions are rendered universal, by means of a general index, which he first introduced into analytical operations (p).

About the same period, several other treatises were also published on particular parts of this science; among the best of which may be reckoned the *Diophantus redivivus* of De Billy, printed in 1670; Kersey's Elements of Algebra, in two volumes folio, 1673; Prestet's *Nouveaux Elémens*

(p) For a farther account of these discoveries, and other particulars relating to this subject, see the corresponding Articles, (v) and (s), Vol. I and II, of the present performance.

des Mathématiques, in two volumes 4to, 1675; and Ozanam's *Élémens d' Algébre*, two volumes 8vo, 1702; all of which may be considered as very ample and ingenious treatises, for that time; particularly as far as relates to the Diophantine analysis, which is here fully explained, and illustrated with a great variety of examples; some of which are of a very difficult nature, requiring all the resources of the art to obtain their solutions.

But the most complete performance of this kind is that contained in the second volume of Euler's Algebra; which work was first published, in the German language, in 1770, and has since been translated both into French and English, in two volumes 8vo, with various notes and additions by the editors; among the latter of which we have a very learned and copious tract of Lagrange on continued fractions, and such parts of the indeterminate analysis as had not been sufficiently treated of by the author; the whole forming one of the most profound treatises on this branch of the subject that has ever yet appeared.

We may here likewise mention the great advantage that mathematical science has derived from the theory of logarithms, which was first reduced to an algebraic form by Halley, and since, in a more perspicuous and satisfactory manner by Lagrange and others; who, by deriving the series, by which these numbers are expressed, from pure analytical principles, have not only rendered the mode of investigating them more conformable to the nature of the subject, but have also simplified their algorithm; and, by that means, made the conception of them, as well as their application, more easy and natural.

The theory of series, also, in all its branches, was cultivated, with great success, during the last two centuries, by John and James Bernoulli, Taylor, Demoivre, Nicole, Stirling, Euler, Lorgna, Simpson, Landen, Waring and others; whose researches have greatly extended and improved this important branch of analysis; but it was chiefly under the hands of Daniel Bernoulli and Euler, that this doctrine was augmented and generalized, by the simple means only of the ordinary algebraical calculus, which they have employed, in many of the most difficult cases, with great skill and address.

It may be likewise farther observed, that the exponential formulæ for the sines and cosines of arcs, which were first given by Demoivre, have greatly contributed to the extension of the analytical part of trigonometry, by abridging its operations and shortening the labour of investigation. A considerable improvement was also made in this branch of the science by Euler, who, instead of using separate letters to denote the sine, cosine, &c. of an arc, or angle, employed the abbreviations *sin a*, *cos a*, *tan a*, &c. which method affords a great relief to the memory, and saves the trouble of repeated reference to the figure to which they belong.

We have here also to mention the highly interesting work lately published by Gauss, entitled *Disqui-*

sitiones Arithmeticæ, which principally treats on the theory of numbers, but contains, also, a complete solution of binomial equations of different orders; by means of which he has been able to inscribe, geometrically, a polygon of 17 sides in a circle, a thing that had hitherto been thought impossible. The theory he has laid down, also leads to a more simple and perspicuous demonstration of the celebrated theorem of Cotes, relating to the division of the circle, than any that has yet been given, as well as to a varety of other matters, of equal curiosity and importance.

Legendre has also lately published a work of great ingenuity and research, on the theory of numbers; in which, among a variety of other things, he shows how to approximate towards the roots of equations in general, by means of continued fractions. We are likewise indebted to Arbogast, for a very profound analytical work, entitled *Du Calcul des Dérivations,* published in 1804; as also to Euler, Waring, Simpson, Bezout and others, who, in their various performances, have all treated on different parts of this subject, and made many important advances in the science.

To these we have likewise farther to add, the two celebrated works of Lagrange, entitled the *Théorie des Fonctions analytiques,* and *Lesçons sur le Calcul des Fonctions;* as also his treatise entitled *De la Résolution des Equations numériques;* in which he has given a method of approximating to the roots of numeral equations, which is more direct and certain than any that has yet appeared; though it must be considered of greater value as a theoretical than a practical rule; being much too complicated for general use.

Having thus given a slight sketch of the discoveries and improvements that have been made in this branch of analysis, from its earliest cultivation in Europe, to the middle of the last century, it will here only be necessary to mention such of the most useful elementary works as are peculiarly calculated for the use of beginners; which are those of Clairaut, Lacroix, Maclaurin, Simpson, Emerson, Saunderson, Dodson, Wood, and my own Introduction to Algebra; which latter performance, it is presumed, from the number of editions it has passed through, has proved of great use in disseminating a knowledge of the science among a class of readers, who, but for such a compendium, would probably never have directed their attention to this subject.

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ERRATA.

Page 53, l. 16, *for* $64x^2$, *read* $64x$.

70, 28, *for* $\sqrt{a+\sqrt{\pm b}}$, and $\sqrt{a-\sqrt{\pm b}}$, *read* $\sqrt[3]{a+\sqrt{\pm b}}$, and $\sqrt[3]{a-\sqrt{\pm b}}$.

126, 13, *for* observing, *read* and observing.

149, 8, *for* $\frac{1}{2}(a-\beta)\sqrt{-3}$, *read* $\frac{1}{2}(\alpha-\beta)\sqrt{-3}$.

151, last l. in the note, *for* $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, *r.* $+\frac{1}{2}-\frac{1}{2}\sqrt{-3}$.

152, 3d l. from the bottom, *for* $\frac{2ar}{3b}\sqrt{\frac{a}{3}}$, *read* $\frac{2ar}{3b}\sqrt{\frac{a}{3}}$ is.

180, 3d l. from the bottom, *for* 3, *read* 4.

193, l. 11, *for* contrary, *read* the same.

204 and 205, *for* multiplication and division of sines, *read* multiplication and division of series.

207, *for* 2, 3, 4, &c. in the denominators, *read* 2A_{ω}, 3A_{ω}, 4A_{ω}, &c.

218, l. 14, *for* number, *read* member; and in the note, *for* differential, *read* différentiel.

224, l. 12, *for* $(n+1)(n+2)(n+3)$, *read* $n(n+1)(n+2)(n+3)$.

258, l. 12, *for* $\frac{29}{16}$, *read* $\frac{29}{16}x^2$.

334, in the note, *for* Sterling, *read* Stirling.

342, l. 8, *for* $\frac{14n^2-14}{4}n$, *read* $\frac{14n^2-14n}{4}$.

391, *for* the last series, at the bottom, *read*

$$\Sigma = a + \frac{bx}{1-x} - \frac{\text{D}'x^2}{(1-x)^2} + \frac{\text{D}''x^3}{(1-x)^3} - \frac{\text{D}'''x^4}{(1-x)^4} + \&c.$$

ALGEBRA.

(A) ALGEBRA is the science which treats of a general method of performing calculations, and resolving mathematical problems, by means of the letters of the alphabet (a).

Its leading rules are the same as those of arithmetic; and the operations to be performed are denoted by the following characters:

+ *plus*, or *more*, the sign of addition; signifying, that the quantities between which it is placed are to be added together.

Thus, $a+b$ shows that the number, or quantity, represented by b, is to be added to that represented by a; and is read a plus b.

− *minus*, or *less*, the sign of subtraction; signifying, that the latter of the two quantities be-

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(a) The science here treated of, which is called, by Newton and others, Universal Arithmetic, differs from that of numbers chiefly in this; that in the latter, every figure has a determinate and individual value peculiar to itself; whereas the algebraic characters, being general, or independent of any particular or partial signification, represent all sorts of numbers, or quantities, according to the nature of the question to which they are applied.

Hence, when any of the operations of addition, multiplication, &c. are to be performed upon numbers, or other magnitudes, which are represented by the letters a, b, c, &c. it is obvious that the results, so obtained, will be general; and that any particular case, of a similar kind, may be readily derived from them tween which it is placed is to be taken from the former.

Thus, $a-b$ shows that the quantity represented by b is to be taken from that represented by a; and is read a minus b.

Also, $a \backsim b$ represents the difference of the two quantities a and b, when it is not known which of them is the greater.

× *into*, the sign of multiplication; signifying that the quantities between which it is placed are to be multiplied together.

Thus, $a \times b$ shows that the quantity represented by a is to be multiplied by that represented by b; and is read a into b.

The multiplication of simple quantities is also frequently denoted by a point; or by joining the letters together in the form of a word.

Thus, $a \times b$, $a.b$, and ab, all signify the product of a and b: also, $3 \times a$, or $3a$, is the product of 3 and a; and is read 3 times a.

÷ *by*, the sign of division; signifying that the former of the two quantities between which it is placed is to be divided by the latter.

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by barely substituting for every letter its real numeral value, and then computing the amount accordingly.

Another advantage also, which arises from this general mode of notation, is, that, while the figures employed in common arithmetic disappear in the course of the operation, the characters used in algebra always retain their original form, so as to show the dependence they have upon each other, in every part of the process; which circumstance, together with that of representing the operations of addition, subtraction, &c. by certain signs, renders the language of this science extremely simple and commodious.

Thus, $a \div b$ shows that the quantity represented by a is to be divided by that represented by b; and is read a by b, or a divided by b.

Division is also frequently denoted by placing one of the two quantities over the other, in the form of a fraction.

Thus, $b \div a$ and $\frac{b}{a}$ both signify the quotient of b divided by a; and $\frac{a-b}{a+c}$ signifies that $a-b$ is to be divided by $a+c$.

$=$ *equal to*, the sign of equality; signifying that the quantities between which it is placed are equal to each other.

Thus, $x = a + b$ shows that the quantity denoted by x is equal to the sum of the quantities a and b; and is read x equal to a plus b.

$\equiv$ *identical to*, or the sign of equivalence, signifying that the expressions between which it is placed are of the same value, for all values of the letters of which they are composed.

Thus, $(x+a) \times (x-a) \equiv x^2 - a^2$, whatever numeral values may be given to the quantities represented by x and a.

$>$ *greater than*, the sign of majority; signifying that the former of the two quantities between which it is placed is greater than the latter.

Thus, $a > b$ shows that the quantity represented by a is greater than that represented by b; and is read a greater than b.

$<$ *less than*, the sign of minority; signifying that the former of the two quantities between which it is placed is less than the latter.

Thus, $a < b$ shows that the quantity represented by a is less than that represented by b; and is read a less than b.

: *as*, or *to*, and : : *so is*, the signs of an equality of ratios; signifying that the quantities between which they are placed are proportional.

Thus, $a : b :: c : d$ denotes that a has the same ratio to b that c has to d, or that a, b, c, d, are proportionals; and is read, as a is to b so is c to d, or a is to b as c is to d.

$\surd$ *the radical sign*, signifying that the quantity before which it is placed is to have some root of it extracted.

Thus, $\sqrt{a}$ is the square root of a; $\sqrt[3]{a}$ is the cube root of a; and $\sqrt[4]{a}$ is the fourth root of a; &c.

The roots of quantities, are also represented by figures placed at the right hand corner of them, in the form of a fraction.

Thus, $a^{\frac{1}{2}}$ is the square root of a; $a^{\frac{1}{3}}$ is the cube root of a; and $a^{\frac{1}{n}}$ is the nth root of a, or a root denoted by any number n.

In like manner, a^2 is the square of a; a^3 is the cube of a; and a^m is the mth power of a, or any power denoted by the number m.

∞ is the sign of infinity, signifying that the quantity standing before it is of an unlimited value, or greater than any quantity that can be assigned.

The coefficient of a quantity is the number or letter which is prefixed to it.

Thus, in the quantities $3b$, $-\frac{2}{3}b$, 3 and $-\frac{2}{3}$ are the coefficients of b; and a is the coefficient of x in the quantity ax.

A quantity without any coefficient prefixed to it is supposed to have 1 or unity; and when a quantity has no sign before it, + is always understood.

Thus, a is the same as $+a$, or $+1a$; and $-a$ is the same as $-a$, or $-1a$.

A term is any part or member of a compound quantity, which is separated from the rest by the signs + or −.

Thus, a and b are the terms of $a+b$; and $3a$, $-2a$, and $+5cd$, are the terms of $3a-2b+5cd$.

In like manner, the terms of a product, fraction, or proportion, are the several parts or quantities of which they are composed.

Thus, a and b are the terms of ab, or of $\frac{a}{b}$; and a, b, c, d, are the terms of the proportion $a : b :: c : d$.

A factor is one of the terms, or multipliers, which form the product of two or more quantities.

Thus, a and b are the factors of ab; also, 2, a, and b^2, are the factors of $2ab^2$; and $a-x$ and $b-x$ are the factors of the product $(a-x)\times(b-x)$.

A composite number, or quantity, is that which is produced by the multiplication of two or more terms or factors.

Thus, 6 is a composite number, formed of the factors 2 and 3, or 2×3; and $3abc$ is a composite quantity, the factors of which are 3, a, b, c.

Like quantities, are those which consist of the same letters or combinations of letters; as a and $3a$, or $5ab$ and $7ab$, or $2a^2b$ and $9a^2b$.

Unlike quantities, are those which consist of different letters, or combinations of letters; as a and b, or $3a$ and a^2, or $5ab^2$ and $7a^2b$.

Given quantities, are such as have known values, and are generally represented by some of the first letters of the alphabet; as a, b, c, d, &c.

Unknown quantities, are such as have no fixed values, and are usually represented by some of the final letters of the alphabet; as x, y, z.

Simple quantities, are those which consist of one term only; as $3a$, $5ab$, $-8a^2b$, &c.

Compound quantities, are those which consist of several terms; as $2a+b$, or $3a-2c$, or $a+2b-3c$, &c.

Positive, or affirmative quantities, are those which are to be added; as a, or $+a$, or $+3ab$, &c.

Negative quantities, are those which are to be subtracted; as $-a$, or $-3ab$, or $-7ab^2$, &c.

Like signs, are such as are all positive, or all negative; as $+$ and $+$, or $-$ and $-$.

Unlike signs, are when some are positive and others negative; as $+$ and $-$, or $-$ and $+$.

A monomial, is a quantity consisting of one term only; as a, $2b$, $-3a^2b$, &c.

A binomial, is a quantity consisting of two terms; as $a+b$, or $a-b$; the latter of which is, also, sometimes called a residual quantity.

A trionomial, is a quantity consisting of three terms, as $a+2b-3c$; a quadrinomial of four, as $a-2b+3c-d$; and a polynomial, or multinomial, is that which has many terms.

The power of a quantity, is its square, cube, biquadrate, &c.; called also its second, third, fourth power, &c.; as a^2, a^3, a^4, &c.

The index, or exponent of a quantity, is the number which denotes its power or root.

Thus, -1 is the index of a^{-1}, 2 is the index of a^2, and $\frac{1}{2}$ of $a^{\frac{1}{2}}$ or $\sqrt{a}$.

When a quantity appears without any index, or exponent, it is always understood to have unity, or 1.

Thus, a is the same as a^1, and $2x$ is the same as $2x^1$; the 1, in such cases, being usually omitted.

A rational quantity, is that which can be expressed in finite terms, or without any radical sign, or fractional index; as a, or $\frac{1}{3}a$, or $5a^2$, &c.

An irrational quantity, or surd, is that which has no exact root, or which can only be expressed by means of the radical sign, or a fractional index; as $\sqrt{2}$ or $2^{\frac{1}{2}}$, $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$, &c.

A square or cube number, &c., is that which has an exact square or cube root, &c.

Thus, 4 and $\frac{9}{16}a^2$ are square numbers; and 64 and $\frac{8}{27}a^3$ are cube numbers, &c.

A measure of any quantity, is that by which it can be divided without leaving a remainder.

Thus, 3 is a measure of 6, $7a$ is a measure of $35a$, and $9ab$ of $27a^2b^2$.

Commensurable quantities, are such as can be each divided by the same quantity, without leaving a remainder.

Thus, 6 and 8, $2\sqrt{2}$ and $3\sqrt{2}$, $5a^2b$ and $7ab^2$, are commensurable quantities; the common divisors being 2, $\sqrt{2}$, and ab.

Incommensurable quantities, are such as have no common measure, or divisor, except unity.

Thus, 15 and 16, $\sqrt{2}$ and $\sqrt{3}$, and $a+b$ and a^2+b^2, are incommensurable quantities.

A multiple of any quantity, is that which is some exact number of times that quantity.

Thus, 12 is a multiple of 4, $15a$ is a multiple of $3a$, and $20a^2b^2$ of $5ab$.

The reciprocal of any quantity, is that quantity inverted, or unity divided by it.

Thus, the reciprocal of a, or $\frac{a}{1}$, is $\frac{1}{a}$; and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

A function of one or more quantities, is an expression into which those quantites enter, in any manner whatever, either mixed, or not, with known quantities (*b*).

Thus, $a-2x$, $ax+3x^2$, $2x-a(a^2+x^2)^{\frac{1}{2}}$, ax^m, a^x, &c., are functions of x; and $axy+bx^2$, $ay+x(ax^2-by^3)^{\frac{1}{2}}$, &c. are functions of x and y.

A vinculum, is a bar —, or parenthesis ( ), made use of to collect several quantities into one.

Thus, $\overline{a+b}\times c$, or $(a+b)c$, denotes that the compound quantity $a+b$ is to be multiplied by the simple quantity c; and $\sqrt{\overline{ab+c^2}}$, or $(ab+c^2)^{\frac{1}{2}}$, is the square root of the compound quantity $ab+c^2$.

(*b*) The term function, which is scarcely to be found in any of our algebraical treatises, was used by the early analysts, to signify the different powers or roots of the same quantity; so that, supposing m to be any whole or fractional number, a^m and x^m would have been called by them functions of a and x; but the phrase is now generally understood in the extended sense above mentioned.

Practical Examples for computing the numeral Values of various Algebraic Expressions, or Combinations of Letters.

Supposing $a=6$, $b=5$, $c=4$, $d=1$, and $e=0$. Then

$$a^2+2ab-c+d=36+60-4+1=93.$$

And

$$2a^3-3a^2b+c^3=432-540+64=-44.$$

And

$$a^2\times\overline{a+b}-2abc=36\times 11-240=156.$$

And

$$3a\sqrt{2ac+c^2}, \text{ or } 3a(2ac+c^2)^{\frac{1}{2}}=18\sqrt{64}=144.$$

And

$$2a\sqrt{b^2-ac}+\sqrt{2ac+c^2}=12\times 1+8=20.$$

And

$$\sqrt{2a^2-\sqrt{2ac+c^2}}=\sqrt{72-\sqrt{64}}=\sqrt{72-8}=\sqrt{64}=8.$$

And

$$\frac{2a+3c}{6d+4e}+\frac{4bc}{\sqrt{2ac+c^2}}=\frac{24}{6}+\frac{80}{8}=4+10=14.$$

Required the numeral values of the following quantities; supposing a, b, c, d, e, to be 6, 5, 4, 1, and 0, respectively, as above.

$$2a^2+3bc-5d$$

$$5a^2b-10ab^2+2e$$

$$7a^2+\overline{b-c}\times\overline{d-e}$$

$$\frac{ab^2}{c}\times d-\frac{a-b}{d}+2a^2e$$

$$3\sqrt{c}+2a\sqrt{2a+b-d}$$

$$a\sqrt{a^2+b^2}-3bc\sqrt{a^2-b^2}$$

$$3a^2b+\sqrt[3]{c^2-\sqrt{2ac+c^2}}.$$

$$\frac{2b+c}{3a-c}-\frac{\sqrt{5b}-3\sqrt{c}+d}{2a+c}$$

ADDITION.

(B) ADDITION is the connecting of quantities together by means of their proper signs, and incorporating such as are like, or that can be united, into one sum; the rule for performing which is commonly divided into the three following cases (*c*).

CASE I.

When the quantities are like, and have like signs.

RULE.

Add all the coefficients of the several quantities together, and to their sum annex the letter or letters belonging to each term, prefixing, when necessary, the common sign.

EXAMPLES.

$3a$	$-3ax$	$2b+3y$
$5a$	$-6ax$	$5b+7y$
a	$-ax$	$b+2y$
$7a$	$-2ax$	$8b+y$
$12a$	$-7ax$	$4b+4y$
$28a$	$-19ax$	$20b+17y$

(*c*) It may here be observed, that the term Addition, which is applied to this rule, in conformity to custom, is too scanty to express the nature of the operations to be performed, which are sometimes those of addition and sometimes subtraction, according as the quantities are positive or negative. It should, therefore, be called by some name signifying incorporation, union, or striking a balance, in which case, the incongruity that now subsists, would be removed.

$$\begin{array}{r} 2ay \\ 5ay \\ 4ay \\ 7ay \\ 16ay \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} -2by^2 \\ -6by^2 \\ -by^2 \\ -8by^2 \\ -by^2 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} a-2x^2 \\ a-6x^2 \\ 4a-x^2 \\ 3a-5x^2 \\ 7a-x^2 \\ \hline \\ \hline \end{array}$$

$$\begin{array}{r} 3ax^2 \\ 2ax^2 \\ 12ax^2 \\ 9ax^2 \\ 10ax^2 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 7x-4y \\ x-8y \\ 3x-y \\ x-3y \\ 4x-y \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 2a+x^2 \\ 3a+x^2 \\ a+2x^2 \\ 9a+3x^2 \\ 4a+x^2 \\ \hline \\ \hline \end{array}$$

CASE II.

When the quantities are like, but have unlike signs.

RULE.

Add all the affirmative coefficients into one sum, and those that are negative into another, when there are several of the same kind; then subtract the least of these sums from the greatest, and to the difference prefix the sign of the greater, annexing the common letter or letters, as before.

The principles upon which the above operations are founded, as well as the demonstrations of all the following rules, are given in the second part of the work; to which the reader, who wishes to obtain a theoretical knowledge of the science here treated of, is referred; where he will find the several corresponding articles designated by the same letters, A, B, C, &c. as in the first, or present part.

EXAMPLES.

$$\begin{array}{rcrcr}
-3a & \qquad & 2a-3x^2 & \qquad & 6x+5ay \\
+7a & & -7a+5x^2 & & -3x+2ay \\
+8a & & -3a+x^2 & & x-6ay \\
-a & & +a-3x^2 & & 2x+ay \\
\hline
+11a & & -7a \quad * & & 6x+2ay \\
\hline
\end{array}$$

$$\begin{array}{rcrcr}
-2a^2 & \qquad & 2ay-7 & \qquad & -3ab+7x \\
-3a^2 & & -ay+8 & & +3ab-10x \\
-8a^2 & & +2ay-9 & & +3ab-6x \\
+10a^2 & & -3ay-11 & & -ab+2x \\
+13a^2 & & +12ay+13 & & +2ab+7x \\
\hline
\\
\hline
\end{array}$$

$$\begin{array}{rcrcr}
-2a\sqrt{x} & \qquad & -6a^2+2b & \qquad & 6ax^2+5x^{\frac{1}{2}} \\
+a\sqrt{x} & & +2a^2-3b & & -2ax^2-6x^{\frac{1}{2}} \\
-3a\sqrt{x} & & -5a^2-8b & & +3ax^2-10x^{\frac{1}{2}} \\
+7a\sqrt{x} & & +4a^2-2b & & -7ax^2+3x^{\frac{1}{2}} \\
-4a\sqrt{x} & & -3a^2+9b & & +ax^2+11x^{\frac{1}{2}} \\
\hline
\\
\hline
\end{array}$$

CASE III.

When the quantities are unlike; or some like and others unlike.

RULE.

Collect all the like quantities together, as in the foregoing cases, and set down those that are unlike, one after another, with their proper signs (*d*).

EXAMPLES.

$$\begin{array}{r} 5xy \\ 4ax \\ -xy \\ -4ax \\ \hline 4xy \\ \hline \end{array} \qquad \begin{array}{r} 2xy-2x^2 \\ 3x^2+xy \\ x^2-xy \\ 4x^2-3xy \\ \hline 6x^2-xy \\ \hline \end{array} \qquad \begin{array}{r} 2ax-120 \\ 3x^2-2ax \\ 5x^2-3x^{\frac{1}{2}} \\ 3\sqrt{x}+100 \\ \hline 8x^2-20 \\ \hline \end{array}$$

$$\begin{array}{r} 2ax^{\frac{1}{2}} \\ -ax^2 \\ +3ax^2 \\ -ax^{\frac{1}{2}} \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 8a^2x^2-3ax \\ 7ax-5xy \\ 9xy-5ax \\ 2a^2x^2+xy \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 10b^2-3a^2x \\ -b^2+2a^2x^2 \\ 50+2a^2x \\ a^2x^2+120 \\ \hline \\ \hline \end{array}$$

$$\begin{array}{r} +3x^2y \\ -2xy^2 \\ -3y^2x \\ -8x^2y \\ +2xy^2 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 2\sqrt{x}-8y \\ 3\sqrt{xy}+10x \\ 2x^2y+5y \\ 12xy-\sqrt{xy} \\ -8y+x^{\frac{1}{2}} \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 2a^2-20\sqrt{ax} \\ x^2-2a^{\frac{1}{2}}x^{\frac{1}{2}} \\ 3a^2-13xy \\ xy+32a^2 \\ 20-65x^2 \\ \hline \\ \hline \end{array}$$

(*d*) When several quantities are to be added together, it is the same thing, in whatever order they are placed. Thus, $a+b-c$, $-c+a+b$, or $a-c+b$, are equivalent expressions; though it is usual, in such cases, to make the leading term positive.

Note. When quantities with literal coefficients are to be added together, it may be done by placing the coefficients, with their proper signs, under a vinculum, or between brackets, and then joining the sum or difference, thus arising, to the common quantity, as below:

$$\begin{array}{r} ax+b \\ cx+d \\ \hline (a+c)x+b+d \end{array} \qquad\qquad \begin{array}{r} ax^2+bx \\ cx^2-dx \\ \hline (a+c)x^2+(b-d)x \end{array}$$

EXAMPLES FOR PRACTICE.

1. It is required to add $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$ together.

2. Add $5x-3a+b+7$ and $-4a-3x+2b-9$ together.

3. Add $2a+3b-4c-9$ and $5a-3b+2c-10$ together.

4. Add $3a+2b-5$, $a+5b-c$, and $6a-2c+3$ together.

5. Add x^3+ax^2+bx+2 and x^3+cx^2+dx-1 together.

SUBTRACTION.

(c) SUBTRACTION is the taking of one quantity from another; or the method of finding the difference between any two quantities of the same kind; which is performed as follows: (*e*)

(*e*) The term subtraction, used for this rule, is liable to the same objection as that for addition; the operations to be performed being of a mixed nature, like those of the former. It

RULE.

Change all the signs (+ and −) of the lower line, or quantities that are to be subtracted, into the contrary signs, or rather conceive them to be so changed, and then collect the terms together, as in the several cases of addition.

EXAMPLES.

$$\begin{array}{r} 5a^2-2b \\ 2a^2+5b \\ \hline 3a^2-7b \\ \hline \end{array} \qquad \begin{array}{r} x^2-2y+3 \\ 4x^2+9y-5 \\ \hline -3x^2-11y+8 \\ \hline \end{array} \qquad \begin{array}{r} 5xy+8x-2 \\ 3xy-8x-7 \\ \hline 2xy+16x+5 \\ \hline \end{array}$$

$$\begin{array}{r} 5xy-18 \\ -xy+12 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 8y^2-2y-5 \\ -y^2+3y+2 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 10-8x-3xy \\ -x+3-xy \\ \hline \\ \hline \end{array}$$

$$\begin{array}{r} -5x^2y-8a \\ +3x^2y-7b \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 4\sqrt{ax}-2x^2y \\ 3\sqrt{ax}-5xy^2 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 5x^2+\sqrt{x}-4y \\ 6x^2-8x-x^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

Note. When the quantities that are to be subtracted have literal coefficients, the operation may be performed by placing the coefficients, with their

---

may also be observed, that, though the rule here given, for subtracting one quantity from another, is universally applicable, it will not always be necessary to have recourse to it; for if $3a$ is to be taken from $7a$, it is plain that the remainder will be $4a$, without any consideration of the change of signs.

proper signs, between brackets, as in addition, and then subjoining the common quantity: thus,

$$\begin{array}{r} ax-b \\ cx-d \\ \hline (a-c)x+d-b \\ \hline \end{array} \qquad\qquad \begin{array}{r} ax^2+bx \\ cx^2-cx \\ \hline (a-c)x^2+(b+e)x \\ \hline \end{array}$$

EXAMPLES FOR PRACTICE.

1. Let $\frac{1}{2}(a-b)$ be subtracted from $\frac{1}{2}(a+b)$.
2. From $3x-2a-b+7$, take $8-3b+a+4x$.
3. From $3a+b+c-2d$, take $b-8c+2d-8$.
4. From $5ab+2b^2-c+bc-b$, take $b^2-2ab+bc$.
5. From ax^3-bx^2+cx-d, take $bx^2+ex-2d$.

MULTIPLICATION.

(D) MULTIPLICATION, or the finding of the product of two or more quantities, is performed in the same manner as in arithmetic; except that it is usual, in this case, to begin the operation at the left hand, and to proceed towards the right, or contrary to the way of multiplying numbers.

The rule is commonly divided into three cases; in each of which, it is necessary to observe, that like signs, in multiplying, produce +, and unlike signs, −.

It is also to be remarked, that powers, or roots of the same quantity, are multiplied together by adding their indices: thus,

$a \times a^2$, or $a^1 \times a^2 = a^3$; $a^2 \times a^3 = a^5$; $a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{5}{6}}$; and $a^m \times a^n = a^{m+n}$.

CASE I.

When the factors are both simple quantities.

RULE.

Multiply the coefficients of the two terms together, and to the product annex all the letters, or their powers, belonging to each, after the manner of a word; and the result, with the proper sign prefixed, will be the product required (*f*).

EXAMPLES.

$12a$	$-2a$	$+5a$	$-9x^2$
$3b$	$+4b$	$-6x$	$-5bx$
$36ab$	$-8ab$	$-30ax$	$+45bx^3$
$7ab$	$-6a^2x$	$-2xy^2$	$-7axy$
$-5ac$	$+5x$	$-xy$	$+6ay$
$3a^2b$	$12a^2x$	$-6xyz$	$-a^2xy$
$2ba^2$	$-2x^2y$	$+ay^2z$	$+2xy^2$

(*f*) When any number of quantities are to be multiplied together, it is the same thing in whatever order they are placed: thus, if ab is to be multiplied by c, the product is either abc, acb, or bca, &c.; though it is usual, in this case, as well as in addition and subtraction, to put them according to their rank in the alphabet. It may here also be observed, in conformity to the rule given above for the signs, that $(+a) \times (+b)$, or $(-a) \times (-b) = +ab$; and $(+a) \times (-b)$, or $(-a) \times (+b) = -ab$.

CASE II.

When one of the factors is a compound quantity.

RULE.

Multiply every term of the compound factor, considered as a multiplicand, separately, by the multiplier, as in the former case; then these products, placed one after another, with their proper signs, will be the whole product required.

EXAMPLES.

$3a-2b$	$6xy-8$	a^2-2x+1
$4a$	$3x$	$4x$
$12a^2-8ab$	$18x^2y-24x$	$4a^2x-8x^2+4x$

$12x-ab$	$35x^2-7a$	$3y^2+5xy-2$
$5a$	$-2x$	xy

$13x^2-a^2b$	$25xy+3a^2$	$3x^2-xy-2y^2$
$-2a$	$13x^2$	$5x^2y$

CASE III.

When both the factors are compound quantities.

RULE.

Multiply every term of the multiplicand separately, by each term of the multiplier, setting down the products one after another, with their proper signs; then add the several lines of products together, and their sum will be the whole product required (g).

EXAMPLES.

$$\begin{array}{l}
x+y \\
x+y \\
\hline
x^2+xy \\
\quad +xy+y^2 \\
\hline
x^2+2xy+y^2 \\
\hline
\end{array}
\qquad
\begin{array}{l}
5x+4y \\
3x-2y \\
\hline
15x^2+12xy \\
\quad -10xy-8y^2 \\
\hline
15x^2+2xy-8y^2 \\
\hline
\end{array}
\qquad
\begin{array}{l}
x^2+xy-y^2 \\
x-y \\
\hline
x^3+x^2y-xy^2 \\
\quad -x^2y-xy^2+y^3 \\
\hline
x^3 \;\; * \;\; -2xy^2+y^3 \\
\hline
\end{array}$$

$$\begin{array}{l}
x+y \\
x-y \\
\hline
x^2+xy \\
\quad -xy-y^2 \\
\hline
x^2 \;\; * \;\; -y^2 \\
\hline
\end{array}
\qquad
\begin{array}{l}
x^2+y \\
x^2+y \\
\hline
x^4+x^2y \\
\quad +x^2y+y^2 \\
\hline
x^4+2x^2y+y^2 \\
\hline
\end{array}
\qquad
\begin{array}{l}
x^2+xy+y^2 \\
x-y \\
\hline
x^3+x^2y+xy^2 \\
\quad -x^2y-xy^2-y^3 \\
\hline
x^3 \;\; * \;\; * \;\; -y^3 \\
\hline
\end{array}$$

(g) Besides the advantages (mentioned in note (a), p. 1) which the algebraic method of notation possesses over that of numbers, it may here be observed, that even in this early part of the science, we are furnished with the means of obtaining several general theorems that could not have been established by the common rules of arithmetic. Thus, it will readily appear, from the two leading examples for practice, given in addition and subtraction, that the greater (a) of any two unequal quantities (a, b) is equal to half their sum *plus* half their difference; and that the less is equal to half their sum *minus* half their difference; that is,

$$a = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b); \text{ and } b = \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b)$$

And, in example 4, of multiplication, given above, it is evident that the product of the sum and difference of any two unequal quantities (a, b) is equal to the difference of their squares; that is,

$$(a+b) \times (a-b) = a^2 - b^2$$

Which are some of the first, or most simple instances of equalities, or rather identities, that occur in this part of the subject.

Note. When several compound quantities are to be multiplied together, the product will be found by multiplying the first by the second, and then that result by the third, and so on to the last factor, as below.

$$\begin{array}{l}
x+a \\
x+b \\
\hline
x^2+ax \\
\quad\ +bx+ab \\
\hline
x^2+(a+b)x+ab \\
x+c \\
\hline
x^3+(a+b)x^2+abx \\
\qquad\quad +cx^2+c(a+b)x+abc \\
\hline
x^3+(a+b+c)x^2+(ab+ac+bc)x+abc
\end{array}$$

$$\begin{array}{l}
x-a \\
x-b \\
\hline
x^2-ax \\
\quad\ -bx+ab \\
\hline
x^2-(a+b)x+ab \\
x-c \\
\hline
x^3-(a+b)x^2+abx \\
\quad -cx^2+c(a+b)x-abc \\
\hline
x^3-(a+b+c)x^2+(ab+ac+bc)x-abc
\end{array}$$

It is also to be observed, that the multiplication of compound quantities is sometimes barely denoted by writing them down, with their proper signs, under a vinculum, without performing the whole operation: as $3ab(a-b)$, or $2a\sqrt{(a^2+b^2)}$ (*h*).

EXAMPLES FOR PRACTICE.

1. Required the product of x^2-xy+y^2 and $x+y$.
2. Required the product of $x^3+x^2y+xy^2+y^3$ and $x-y$.
3. Required the product of x^2+xy+y^2 and x^2-xy+y^2.
4. Required the product of $3x^2-2xy+5$ and $x^2+2xy-3$.
5. Required the product of $2a^2-3ax+4x^2$ and $5a^2-6ax-2x^2$.
6. Required the product of $3x^3+2x^2y^2+3y^3$ and $2x^3-3x^2y^2+5y^3$.
7. Required the product of x^3-ax^2+bx-c and x^2-dx+e.

DIVISION.

(E) DIVISION is the converse of multiplication, and is performed like that of numbers; the rule being usually divided into three cases, in each of

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(*h*) This method of representing the multiplication of compound quantities, by barely indicating the operation that is to be performed on them, is often preferable to that of executing the entire process; particularly when the product of two or more factors is to be divided by some other quantity; because, in this case, any term that is common to both the divisor and dividend, may be more readily suppressed, as will be evident, from a variety of instances, in the following part of the work.

which like signs give + in the quotient, and unlike signs −, as in finding their products (*i*).

It is here also to be observed, that powers and roots of the same quantity, are divided by subtracting the index of the divisor from that of the dividend: thus, $a^3 \div a^2 = a$; $a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{1}{6}}$; and $a^m \div a^n = a^{m-n}$.

CASE I.

When the divisor and dividend are both simple quantities.

RULE.

Set the dividend over the divisor, in the manner of a fraction, and reduce it to its simplest form, by cancelling the letters and figures that are common to each term.

EXAMPLES.

$6ab \div 2a$, or $\frac{6ab}{2a} = 3b$; and $12ax^2 \div 3x$, or $\frac{12ax^2}{3x} = 4ax$; $a \div a$, or $\frac{a}{a} = 1$; and $a \div -a$, or $\frac{a}{-a} = -1$. Also $-2a \div 3a$, or $\frac{-2a}{3a} = -\frac{2}{3}$.

1. Divide $16x^2$ by $8x$, and $12a^2x^2$ by $-8a^2x$.
2. Divide $-15ay^2$ by $3ay$, and $-18ax^2y$ by $-8ax$.

(*i*) According to the rule here given for the signs, it follows, that $\frac{+ab}{+b} = +a$, $\frac{-ab}{-b} = +a$, $\frac{-ab}{+b} = -a$, $\frac{+ab}{-b} = -a$, as will readily appear by multiplying the quotient by the divisor; the signs of the products being then the same as would take place in the former rule.

CASE II.

When the divisor is a simple quantity, and the dividend a compound one.

RULE.

Divide each term of the dividend by the divisor, as in the former case; setting down such as will not divide in the simplest form they will admit of.

EXAMPLES.

$(ab+b^2)\div 2b$, or $\frac{ab+b^2}{2b}=\frac{1}{2}a+\frac{1}{2}b=\frac{a+b}{2}$.

$(10ab-15ax)\div 5a$, or $\frac{10ab-15ax}{5a}=2b-3x$.

$(30ax-48x^2)\div 6x$, or $\frac{30ax-48x^2}{6x}=5a-8x$.

1. Let $3x^3+6x^2+3ax-15x$ be divided by $3x$.
2. Let $3abc+12abx-9a^2b$ be divided by $3ab$.
3. Let $40a^3b^3+60a^2b^2-17ab$ be divided by $-ab$.
4. Let $15a^2bc-12acx^2+5ad^2$ be divided by $-5ac$.

CASE III.

When the divisor and dividend are both compound quantities.

RULE.

Set them down in the same manner as in division of numbers, ranging the terms of each of them so, that the higher powers of one of the letters may stand before the lower. Then divide the first term of the dividend by the first term of the divisor, and set the result in the quotient, with its proper sign, or simply by itself, if it be affirmative. This being done, multiply the whole divisor by the term thus found; and, having subtracted the result from the dividend, bring down as many terms to the remainder as are requisite for the next operation, which perform as before: and so on, till the work is finished, as in common arithmetic.

EXAMPLES.

$$\begin{array}{l}
x+y)x^2+2xy+y^2(x+y \\
\qquad \underline{x^2+xy} \\
\qquad\qquad xy+y^2 \\
\qquad\qquad \underline{xy+y^2} \\
\qquad\qquad\quad *
\end{array}$$

$$\begin{array}{l}
a+x)a^3+5a^2x+5ax^2+x^3(a^2+4ax+x^2 \\
\qquad \underline{a^3+a^2x} \\
\qquad\qquad 4a^2x+5ax^2 \\
\qquad\qquad \underline{4a^2x+4ax^2} \\
\qquad\qquad\qquad ax^2+x^3 \\
\qquad\qquad\qquad \underline{ax^2+x^3} \\
\qquad\qquad\qquad\quad *
\end{array}$$

$$\begin{array}{l}
x-3)x^3-9x^2+27x-27(x^2-6x+9 \\
\qquad \underline{x^3-3x^2} \\
\qquad\quad -6x^2+27x \\
\qquad\quad \underline{-6x^2+18x} \\
\qquad\qquad\qquad 9x-27 \\
\qquad\qquad\qquad \underline{9x-27} \\
\qquad\qquad\qquad\quad *
\end{array}$$

$$\begin{array}{l}
2x^2 - 3ax + a^2)4x^4 - 9a^2x^2 + 6a^3x - a^4(2x^2 + 3ax - a^2 \\
\qquad\qquad\quad 4x^4 - 6ax^3 + 2a^2x^2 \\
\hline
\qquad\qquad\qquad\quad 6ax^3 - 11a^2x^2 + 6a^3x \\
\qquad\qquad\qquad\quad 6ax^3 - 9a^2x^2 + 3a^3x \\
\hline
\qquad\qquad\qquad\qquad\quad - 2a^2x^2 + 3a^3x - a^4 \\
\qquad\qquad\qquad\qquad\quad - 2a^2x^2 + 3a^3x - a^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad *
\end{array}$$

Note. 1. $x^m + a^m$ is divisible by $x + a$, when m is an odd number; but in no case by $x - a$: $x^m - a^m$ is divisible by $x + a$, when m is an even number, and by $x - a$ when m is any whole number whatever. Thus (*k*),

$$\frac{x^2 - a^2}{x + a} = x - a,$$

$$\frac{x^2 - a^2}{x - a} = x + a.$$

$$\frac{x^3 + a^3}{x + a} = x^2 - ax + a^2.$$

$$\frac{x^3 - a^3}{x - a} = x^2 + ax + a^2.$$

$$\frac{x^4 - a^4}{x + a} = x^3 - ax^2 + a^2x - a^3.$$

$$\frac{x^4 - a^4}{x - a} = x^3 + ax^2 + a^2x + a^3.$$

$$\frac{x^5 + a^5}{x + a} = x^4 - ax^3 + a^2x^2 - a^3x + a^4.$$

$$\frac{x^5 - a^5}{x - a} = x^4 + ax^3 + a^2x^2 + a^3x + a^4.$$

&c. &c.

(*k*) The quotient arising from dividing $x^m - a^m$ by $x - a$, when m is a whole number, may be exhibited, in general terms, as follows:

$$\frac{x^m - a^m}{x - a} = x^{m-1} + ax^{m-2} + a^2x^{m-3} \; . \; . \; . \; . \; + a^{m-2}x + a^{m-1}.$$

2. If the divisor be not exactly contained in the dividend, the quantity that remains after the division is finished, must be placed over the divisor, at the end of the quotient, in the form of a fraction: thus (*l*),

$$\begin{array}{l}
a+x)a^3-x^3(a^2-ax+x^2-\frac{2x^3}{a+x} \\
\quad\underline{a^3+a^2x} \\
\qquad -a^2x-x^3 \\
\qquad \underline{-a^2x-ax^2} \\
\qquad\qquad ax^2-x^3 \\
\qquad\qquad \underline{ax^2+x^3} \\
\qquad\qquad\quad -2x^3
\end{array}$$

$$\begin{array}{l}
x+y)x^4+y^4(x^3-x^2y+xy^2-y^3+\frac{2y^4}{x+y} \\
\quad\underline{x^4+x^3y} \\
\qquad -x^3y+y^4 \\
\qquad \underline{-x^3y-x^2y^2} \\
\qquad\qquad x^2y^2+y^4 \\
\qquad\qquad \underline{x^2y^2+xy^3} \\
\qquad\qquad\quad -xy^3+y^4 \\
\qquad\qquad\quad \underline{-xy^3-y^4} \\
\qquad\qquad\qquad 2y^4
\end{array}$$

(*l*) In the case here given, the operation of division may be considered as terminated, when the highest power of the letter, in the first or leading term of the remainder, by which the process is regulated, is less than the first term of the divisor; as the remaining part of the quotient, after this, would become fractional.

3. The division of quantities may also be sometimes carried on, *ad infinitum*, like a decimal fraction; in which case, a few of the leading terms of the quotient will, generally, be sufficient to indicate the rest, without its being necessary to continue the operation: thus,

$$\begin{array}{l}
a+x)a\;\ldots\ldots\;(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\&c.\\
\qquad\quad a+x\\
\qquad\quad \overline{}\\
\qquad\quad -x\\
\qquad\quad -x-\frac{x^2}{a}\\
\qquad\quad \overline{\phantom{-x-\frac{x^2}{a}}}\\
\qquad\qquad\qquad \frac{x^2}{a}\\
\qquad\qquad\qquad \frac{x^2}{a}+\frac{x^3}{a^2}\\
\qquad\qquad\qquad \overline{\phantom{\frac{x^2}{a}+\frac{x^3}{a^2}}}\\
\qquad\qquad\qquad\quad -\frac{x^3}{a^2}\\
\qquad\qquad\qquad\quad -\frac{x^3}{a^2}-\frac{x^4}{a^3}\\
\qquad\qquad\qquad\quad \overline{\phantom{-\frac{x^3}{a^2}-\frac{x^4}{a^3}}}\\
\qquad\qquad\qquad\qquad\quad \frac{x^4}{a^3}
\end{array}$$

Also, $\frac{a}{a-x}=1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\frac{x^4}{a^4}+$ &c.

Where the law, by which either of these series may be continued at pleasure, is obvious.

Examples for Practice.

1. Let $a^2-2ax+x^2$ be divided by $a-x$.
2. Let $x^3-3ax^2+3a^2x-a^3$ be divided by $x-a$.
3. Let $a^3+5a^2x+5ax^2+x^3$ be divided by $a+x$.
4. Let $2y^3-19y^2+26y-17$ be divided by $y-8$.

5. Let x^5+1 be divided by $x+1$, and x^6-1 by $x-1$.

6. Let $48x^3-76ax^2-64a^2x+105a^3$ be divided by $2x-3a$.

7. Let $4x^4-9x^2+6x-3$ be divided by $2x^2+3x-1$.

8. Let $x^4-a^2x^2+2a^3x-a^4$ be divided by x^2-ax+a^2.

9. Let $6x^4-96$ be divided by $3x-6$, and a^6-x^6 by $a-x$.

10. Let b^4-3y^4 be divided by $b-y$, and $a^4+4a^2b+3b^4$ by $a+2b$.

11. Let x^2+ax+b be divided by $x+p$, and x^3-px^2+qx-r by $x-q$.

OF ALGEBRAIC FRACTIONS.

(F) Algebraic fractions have the same names and rules of operation as numeral fractions in common arithmetic; and the methods of reducing them, in either of these branches, to their most convenient forms, are as follows:

CASE I.

To find the greatest common measure of the terms of a fraction.

Rule.

Arrange the two quantities according to the order of their powers, and divide that which is of the highest dimensions by the other, having first expunged any factor, that may be contained in all the terms of the divisor, without being common to those of the dividend; then divide this divisor by the remainder, simplified, if necessary, as before;

and so on, for each successive remainder and its preceding divisor, till nothing remains, when the divisor last used will be the greatest common measure required.

When any of the divisors, in the course of the operation, become negative, they may have their signs changed, or be taken affirmatively, without altering the truth of the result; and if the first term of a divisor should not be exactly contained in the first term of the dividend, the several terms of the latter may be multiplied by any number, or quantity, that will render the division complete (*m*).

EXAMPLES.

1. Required the greatest common measure of the fraction $\frac{x^4-1}{x^5+x^3}$.

$$\begin{array}{r|l}
x^4 - 1)\,x^5 + x^3\,(x & \\
x^5 - x & \\
\cline{1-1}
x^3 + x & \\
\text{or } x^2 + 1 & x^4 - 1\,(x^2 - 1 \\
 & x^4 + x^2 \\
\cline{2-2}
 & -x^2 - 1 \\
 & -x^2 - 1 \\
\cline{2-2}
 & *
\end{array}$$

Where x^2+1 is the common measure required.

(*m*) In finding the greatest common measure of two quantities, either of them may be multiplied, or divided, by any quantity, which is not a divisor of the other, or that contains no factor which is common to them both, without in any respect changing the result.

It may here, also, be farther added, that the common measure, or divisor, of any number of quantities may be determined in a

2. Required the greatest common measure of $\frac{x^3-b^2x}{x^2+2bx+b^2}$.

$$\begin{array}{l}
x^2+2bx+b^2)\,x^3-b^2x\,(x \\
\qquad\qquad\quad x^3+2bx^2+b^2x \\
\qquad\qquad\qquad -2bx^2-2b^2x \\
\qquad\qquad\qquad \text{or } x+b \;\big|\; x^2+2bx+b^2\,(x+b \\
\qquad\qquad\qquad\qquad\qquad\quad x^2+bx \\
\qquad\qquad\qquad\qquad\qquad\qquad bx+b^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad bx+b^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad *
\end{array}$$

Where $x+b$ is the common measure required.

3. Required the greatest common measure of $\frac{3a^2-2a-1}{4a^3-2a^2-3a+1}$.

$$\begin{array}{l}
3a^2-2a-1)\,4a^3-2a^2-3a+1 \\
\qquad\qquad\quad 3 \\
\qquad\qquad\quad 12a^3-6a^2-9a+3\,(4a \\
\qquad\qquad\quad 12a^3-8a^2-4a \\
\qquad\qquad\qquad 2a^2-5a+3)\,3a^2-2a-1 \\
\qquad\qquad\qquad\qquad\qquad\quad 2 \\
\qquad\qquad\qquad\qquad\qquad\quad 6a^2-4a-2\,(3 \\
\qquad\qquad\qquad\qquad\qquad\quad 6a^2-15a+9 \\
\qquad\qquad\qquad\qquad\qquad\qquad 11a-11 \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{or } a-1 \;\big|\; 2a^2-5a+3\,(2a-3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 2a^2-2a \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -3a+3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -3a+3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad *
\end{array}$$

Where $a-1$ is the common measure required.

---

similar manner to that given above, by first finding the common measure of two of them, and then of that common measure and a third; and so on to the last.

4. It is required to find the greatest common measure of $\frac{x^2-1}{ax+a}$.

5. It is required to find the greatest common measure of $\frac{x^3-a^3}{x^4-a^4}$.

6. Required the greatest common measure of $\frac{a^2-ab-2b^2}{a^2-3ab+2b^2}$.

7. Required the greatest common measure of $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$.

8. Required the greatest common measure of $\frac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$.

9. Required the greatest common measure of $\frac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b+2ab^3+b^4}$.

10. Required the greatest common measure of $\frac{7a^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^2}$.

CASE II.

To reduce fractions to their lowest or most simple terms.

RULE.

Divide each of the terms of the fraction by any number, or quantity, that will divide them without leaving a remainder; or find their greatest common measure, as in the last rule, by which divide both the numerator and denominator, and it will give the fraction required.

EXAMPLES.

1. Reduce $\frac{a^2bc}{5a^2b^2}$ and $\frac{ax^2}{ax+x^2}$ to their lowest terms.

$\frac{a^2bc}{5a^2b^2}=\frac{c}{5b}$ Ans. And $\frac{ax^2}{ax+x^2}=\frac{x}{a+x}$ Ans.

2. It is required to reduce $\frac{cx+x^2}{a^2c+a^2x}$ to its lowest terms.

$$\begin{array}{r|l} cx+x^2 & a^2c+a^2x \\ \text{or } c+x & a^2c+a^2x(a^2 \\ & \underline{a^2c+a^2x} \\ & \qquad * \end{array}$$

Therefore $c+x$ is the greatest common measure; and $c+x)\frac{cx+x^2}{a^2c+a^2x}=\frac{x}{a^2}$ the fraction required.

3. It is required to reduce $\frac{x^3-b^2x}{x^2+2bx+b^2}$ to its least terms.

$$\begin{array}{rl} x^2+2bx+b^2)x^3-b^2x(x & \\ \underline{x^3+2bx^2+b^2x} & \\ -2bx^2-2b^2x & \Big| \\ \text{or } x+b & \Big|\; x^2+2bx+b^2(x+b \\ & \quad \underline{x^2+bx} \\ & \qquad bx+b^2 \\ & \qquad \underline{bx+b^2} \\ & \qquad\quad * \end{array}$$

Whence $x+b$ is the greatest common measure; and $x+b)\frac{x^3-b^2x}{x^2+2bx+b^2}=\frac{x^2-bx}{x+b}$ the fraction required.

4. It is required to reduce $\frac{x^4-a^4}{x^5-a^2x^3}$ to its least terms.

5. It is required to reduce $\frac{x^2-a^2}{x^4-a^4}$ to its least terms.

6. It is required to reduce $\frac{a^6-b^6}{a^4-b^4}$ to its least terms.

7. It is required to reduce $\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$ to its lowest terms.

8. It is required to reduce $\frac{2x^3-16x-6}{3x^3-24x-9}$ to its lowest term.

9. It is required to reduce $\frac{48x^3+16x^2-15}{24x^3-22x^2+17x-5}$ to its lowest terms.

10. It is required to reduce $\frac{20x^4+x^2-1}{25x^4+5x^3-x-1}$ to its lowest terms.

11. It is required to reduce $\frac{9x^5-x^3-4x^2+1}{15x^4-2x^3-13x^2-2x+2}$ to its lowest terms.

CASE III.

To reduce a mixed quantity to an improper fraction.

RULE.

Multiply the integral part by the denominator of the fraction, and to the product add the numerator, when it is affirmative, or subtract it when negative; then the result, placed over the denominator, will give the improper fraction required(*n*).

EXAMPLES.

1. Reduce $3\frac{2}{5}$ and $a-\frac{b}{c}$ to improper fractions.

$$3\tfrac{2}{5}=\frac{3\times 5+2}{5}=\frac{15+2}{5}=\frac{17}{5}\text{ Ans.}$$

$$a-\frac{b}{c}=\frac{a\times c-b}{c}=\frac{ac-b}{c}\text{ Ans.}$$

(*n*) If the signs of both the numerator and denominator of any fraction be changed, it will not alter its value: thus,

$$\frac{-a}{-b}=\frac{+a}{+b}, \text{ or } \frac{a}{b}; \text{ and } \frac{a-b}{c-d}=\frac{b-a}{d-c}.$$

And if a fraction has the negative sign before it, the form of it may be changed, without altering its value, by making the numerator only negative: thus, $-\frac{a}{b}=\frac{-a}{b}$.

2. Reduce $x+\frac{a}{x}$ and $x-\frac{a^2-x^2}{x}$ to improper fractions.

$$x+\frac{a}{x}=\frac{x\times x+a}{x}=\frac{x^2+a}{x} \text{ Ans.}$$

$$x-\frac{a^2-x^2}{x}=\frac{x^2-a^2+x^2}{x}=\frac{2x^2-a^2}{x} \text{ Ans.}$$

3. Let $15\frac{2}{3}$ be reduced to an improper fraction.

4. Let $1-\frac{2x}{a}$ be reduced to an improper fraction.

5. Let $x-\frac{ax+x^2}{2a}$ be reduced to an improper fraction.

6. Let $5+\frac{2x-7}{3x}$ be reduced to an improper fraction.

7. Let $1-\frac{x-a-1}{a}$ be reduced to an improper fraction.

8. Let $1+2x-\frac{x-3}{5x}$ be reduced to an improper fraction.

CASE IV.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

Divide the numerator by the denominator, for the integral part, and place the remainder, if any, over the denominator, for the fractional part; then the two, joined together, with the proper sign between them, will give the mixed quantity required.

EXAMPLES.

1. Reduce $\frac{27}{5}$ and $\frac{ax+a^2}{a}$ to mixed quantities.

$$\frac{27}{5}=27\div 5=5\tfrac{2}{5} \text{ Ans.}$$

$$\frac{ax+a^2}{x}=(ax+a^2)\div x=a+\frac{a^2}{x} \text{ Ans.}$$

2. It is required to reduce the fraction $\frac{ax-x^2}{x}$ to a whole quantity.

3. It is required to reduce the fraction $\frac{ab-2a^2}{b}$ to a mixed quantity.

4. It is required to reduce the fraction $\frac{a^2+x^2}{a-x}$ to a mixed quantity.

5. It is required to reduce the fraction $\frac{x^3-y^3}{x-y}$ to a whole quantity.

6. It is required to reduce the fraction $\frac{10x^2-5x+3}{5x}$ to a mixed quantity.

CASE V.

To reduce fractions to other equivalent ones, that shall have a common denominator.

RULE.

Multiply each of the numerators, separately, into all the denominators, except its own, for the new numerators, and all the denominators together for a common denominator (o).

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to fractions that shall have a common denominator.

(o) It may here be remarked, that if the numerator and denominator of a fraction be either both multiplied, or both divided, by the same number, or letter, its value will not be altered: thus, $\frac{2}{3}=\frac{2\times 3}{3\times 3}=\frac{6}{9}$, and $\frac{3}{12}=\frac{3\div 3}{12\div 3}=\frac{1}{4}$; or $\frac{a}{b}=\frac{ac}{bc}$, and $\frac{ab}{bc}=\frac{a}{c}$, which method is often of great use in reducing fractions more readily to a common denominator.

$$\left.\begin{array}{l} a \times c = ac \\ b \times b = b^2 \end{array}\right\} \text{ the new numerators.}$$

$$\overline{b \times c = bc} \quad \text{the common denominator.}$$

Whence $\frac{a}{b}$ and $\frac{b}{c} = \frac{ac}{bc}$ and $\frac{b^2}{bc}$, the fractions required.

2. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions having a common denominator.

3. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to equivalent fractions having a common denominator.

4. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d, to equivalent fractions having a common denominator.

5. Reduce $\frac{3}{4}$, $\frac{2x}{3}$, and $a+\frac{4x}{5}$, to fractions having a common denominator.

6. Reduce $\frac{a}{2}$, $\frac{3x}{7}$, and $\frac{a+x}{a-x}$, to fractions having a common denominator.

CASE VI.

To add fractional quantities together.

RULE.

Reduce the fractions, if necessary, to a common denominator; then add all the numerators together, and under their sum put the common denominator, and it will give the sum of the fractions required (*p*).

(*p*) In the adding or subtracting of mixed quantities, it is best to bring the fractional parts only to a common denominator, and to affix their sum or difference to the sum or difference of the integral parts, interposing the proper sign.

EXAMPLES.

1. It is required to find the sum of $\frac{x}{2}$ and $\frac{x}{3}$.

$$\left.\begin{array}{l} x \times 3 = 3x \\ x \times 2 = 2x \end{array}\right\} \text{the numerators.}$$

$$2 \times 3 = 6 \quad \text{the common denominator.}$$

Whence $\frac{3x}{6} + \frac{2x}{6} = \frac{5x}{6}$, the sum required.

2. It is required to find the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

$$\left.\begin{array}{l} a \times d \times f = adf \\ c \times b \times f = cbf \\ e \times b \times d = ebd \end{array}\right\} \text{the numerators.}$$

$$b \times d \times f = bdf \text{ the common denominator.}$$

Whence $\frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf}$ the sum.

3. It is required to find the sum of $a - \frac{3x^2}{b}$ and $b + \frac{2ax}{c}$.

$$\left.\begin{array}{l} 3x^2 \times c = 3cx^2 \\ 2ax \times b = 2abx \end{array}\right\} \text{the numerators.}$$

$$b \times c = bc \quad \text{the common denominator.}$$

Whence $a - \frac{3cx^2}{bc} + b + \frac{2abx}{bc} = a + b + \frac{2abx - 3cx^2}{bc}$ the sum.

4. It is required to find the sum of $\frac{3x}{2a}$ and $\frac{x}{5}$

5. It is required to find the sum of $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$

6. It is required to find the sum of $\frac{4x}{7}$ and $\frac{x-2}{5}$

7. It is required to find the sum of $5x$, $\frac{2a}{3x^2}$, and $\frac{a+2x}{4x}$

8. It is required to find the sum of $3a+\frac{2x}{5}$ and $a-\frac{8x}{9}$.

9. It is required to find the sum of a, $\frac{a}{a-x}$, and $\frac{a-x}{a}$.

10. It is required to find the sum of $x+\frac{x-2}{3}$ and $4x-\frac{2x-3}{5x}$.

11. It is required to find the sum of $\frac{a}{c}$, $\frac{a-b}{c+d}$, and $\frac{a+b}{c-d}$.

CASE VII.

To subtract one fractional quantity from another.

RULE.

Reduce the fractions to a common denominator, if necessary, as in addition; then subtract the less numerator from the greater, and under the difference write the common denominator, and it will give the difference of the fractions required.

EXAMPLES.

1. It is required to find the difference of $\frac{2x}{3}$ and $\frac{3x}{5}$.

$$\left.\begin{array}{l} 2x\times 5=10x \\ 3x\times 3=9x \end{array}\right\} \text{the numerators.}$$

$$3\times 5=15 \text{ the common denominator.}$$

Whence $\frac{10x}{15}-\frac{9x}{15}=\frac{x}{15}$, the difference required.

2. It is required to find the difference of $\frac{x-a}{2b}$ and $\frac{2a-4x}{3c}$.

$$\left.\begin{array}{l}(x-a)\times 3c=3cx-3ac\\(2a-4x)\times 2b=4ab-8bx\end{array}\right\}\text{ the numerators.}$$

$$2b\times 3c=6bc \text{ the common denominator.}$$

Whence $\frac{3cx-3ac}{6bc}-\frac{4ab-8bx}{6bc}=\frac{3cx-3ac-4ab+8bx}{6bc}$, the difference required.

3. It is required to find the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$

4. It is required to find the difference of $5y$ and $\frac{3y}{8}$.

5. It is required to find the difference of $\frac{ax}{b-c}$ and $\frac{ax}{b+c}$

6. It is required to find the difference of $2x+\frac{2x+7}{8}$ and $x-\frac{5x-6}{21}$

7. It is required to find the difference of $x-\frac{x-a}{c}$ and $3x+\frac{x}{2b}$.

8. It is required to find the difference of $a+\frac{a-x}{a(a+x)}$ and $\frac{a+x}{a(a-x)}$

CASE VIII.

To multiply fractional quantities together.

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator; and the former of these, being placed over the latter, will give the product of the fractions, as required (*q*).

(*q*) When the numerator of one of the fractions to be multi-

EXAMPLES.

1. It is required to find the product of $\frac{x}{6}$ and $\frac{2x}{9}$. $\frac{x \times 2x}{6 \times 9} = \frac{2x^2}{54} = \frac{x^2}{27}$ the product required.

2. It is required to find the product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$. $\frac{x \times 4x \times 10x}{2 \times 5 \times 21} = \frac{40x^3}{210} = \frac{4x^3}{21}$ the product.

3. It is required to find the product of $\frac{x}{a}$ and $\frac{a+x}{a-x}$. $\frac{x \times (a+x)}{a \times (a-x)} = \frac{x^2+ax}{a^2-ax}$ the product.

4. It is required to find the product of $\frac{3x}{2}$ and $\frac{5x}{3b}$

5. It is required to find the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$

6. It is required to find the product of $\frac{2x}{3}$, $\frac{4x^2}{7}$, and $\frac{a}{a+x}$

---

plied, and the denominator of the other, can be divided by some quantity which is common to each of them, the quotients may be used instead of the fractions themselves.

Also, when a fraction is to be multiplied by an integer, it is the same thing whether the numerator be multiplied by it, or the denominator divided by it.

Or, if an integer is to be multiplied by a fraction, or a fraction by an integer, the integer may be considered as having unity for its denominator, and the two be then multiplied together as usual: thus,

$$(a+x) \times \frac{2b}{c} = \frac{a+x}{1} \times \frac{2b}{c} = \frac{2ab+2bx}{c}.$$

7. It is required to find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{5ac}{2b}$.

8. Required the product of $3x$, $\frac{x+1}{2a}$, and $\frac{x-1}{a+b}$.

9. Required the product of $2a+\frac{bx}{a}$ and $3a-\frac{b}{ax}$.

10. Required the product of $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$, and $a+\frac{ax}{a-x}$.

CASE IX.

To divide one fractional quantity by another.

Rule.

Multiply the denominator of the divisor by the numerator of the dividend, for the numerator; and the numerator of the divisor by the denominator of the dividend, for the denominator. Or, which is more convenient in practice, multiply the dividend by the reciprocal of the divisor, and the product will be the quotient required (*r*).

Examples.

1. It is required to divide $\frac{x}{3}$ by $\frac{2x}{9}$

$$\frac{x}{3}\div\frac{2x}{9}=\frac{x}{3}\times\frac{9}{2x}=\frac{9x}{6x}=\frac{3}{2}=1\tfrac{1}{2}\text{ Ans.}$$

(*r*) When a fraction is to be divided by an integer, it is the same thing whether the numerator be divided by it, or the denominator multiplied by it.

Also when the two numerators, or the two denominators, can be divided by some common quantity, that quantity may be thrown out of each, and the quotients used instead of the fractions first proposed.

2. It is required to divide $\frac{2a}{b}$ by $\frac{4c}{d}$

$$\text{Here } \frac{2a}{b} \times \frac{d}{4c} = \frac{2ad}{4bc} = \frac{ad}{2bc} \text{ Ans.}$$

3. It is required to divide $\frac{x+a}{x-b}$ by $\frac{x+b}{5x+a}$

$$\frac{x+a}{x-b} \times \frac{5x+a}{x+b} = \frac{5x^2+6ax+a^2}{x^2-b^2} \text{ Ans.}$$

4. It is required to divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$

$$\frac{2x^2}{a^3+x^3} \times \frac{x+a}{x} = \frac{2x^2(x+a)}{x(a^3+x^3)} = \frac{2x}{x^2-ax+a^2} \text{ Ans.}$$

5. It is required to divide $\frac{7x}{5}$ by $\frac{3}{x}$

6. It is required to divide $\frac{4x^2}{7}$ by $5x$

7. It is required to divide $\frac{x+1}{6}$ by $\frac{2x}{3}$

8. It is required to divide $\frac{x}{1-x}$ by $\frac{x}{5}$

9. It is required to divide $\frac{x-b}{7cd}$ by $\frac{3cx}{4d}$

10. It is required to divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$

11. It is required to divide $x+\frac{2x}{x-3}$ by $x-\frac{2x}{x-3}$

12. Divide $\frac{x^4-a^4}{x^3+a^3}$ by $\frac{x-a}{x+a}$; and $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$

INVOLUTION.

(G) **Involution** is the raising of powers from any proposed root; or the method of finding the square, cube, biquadrate, &c, of any given quantity (*s*).

(*s*) In the denominations of powers, considerations peculiar to Geometry have led us to call the second and third powers, the

RULE I.

Multiply the quantity into itself as many times as is denoted by the index of the power less one, and the last product will be the power required.

Or, multiply the index of the quantity by the index of the power to which it is to be raised, and the result will be the same as before.

Note. When the sign of the root is +, all the powers of it will be +; and when the sign is −, all the even powers will be +, and the odd powers −: as is evident from multiplication (*t*).

EXAMPLES.

a, the root.	a^2 the root.
a^2 = square.	a^4 = square.
a^3 = cube.	a^6 = cube.
a^4 = 4th power.	a^8 = 4th power.
a^5 = 5th power.	a^{10} = 5th power.
&c.	&c.

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square and cube; but when they are of a higher kind, they are usually mentioned by the terms fourth power, fifth power, and so on, according to the index by which they are denoted.

(*t*) Any power of the product of two or more quantities is equal to the same power of each of the factors multiplied together: thus,

$$(ab)^m = a^m \times b^m; \text{ and } (abc)^m = a^m \times b^m \times c^m.$$

And any power of a fraction is equal to the same power of the numerator divided by the like power of the denominator: thus,

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

Also, a^m raised to the nth power is a^{mn}; and $-a^m$ raised to the nth power is $\pm a^{mn}$, according as n is an even or an odd number.

$-3a$ the root.
$+9a^2 =$ square.
$-27a^3 =$ cube.
$+81a^4 =$ 4th power.
$-243a^5 =$ 5th power.
&c.

$-2ax^2$ the root.
$4a^2x^4 =$ square.
$-8a^3x^6 =$ cube.
$+16a^4x^8 =$ 4th power.
$-32a^5x^{10} =$ 5th power.
&c.

$\frac{x}{a}$ the root.
$\frac{x^2}{a^2} =$ square.
$\frac{x^3}{a^3} =$ cube.
$\frac{x^4}{a^4} =$ 4th power.
&c.

$-\frac{2ax^2}{3b}$ the root.
$+\frac{4a^2x^4}{9b^2} =$ square.
$-\frac{8a^3x^6}{27b^3} =$ cube.
$+\frac{16a^4x^8}{81b^4} =$ 4th power.
&c.

$x - a$ the root.

$$\begin{array}{l}
x - a \\
x - a \\
\hline
x^2 - ax \\
\quad - ax + a^2 \\
\hline
x^2 - 2ax + a^2 \text{ square.} \\
x - a \\
\hline
x^3 - 2ax^2 + a^2x \\
\quad - ax^2 + 2a^2x - a^3 \\
\hline
x^3 - 3ax^2 + 3a^2x - a^3 \text{ cube.}
\end{array}$$

&c.

$x + a$ the root.

$$\begin{array}{l}
x + a \\
x + a \\
\hline
x^2 + ax \\
\quad + ax + a^2 \\
\hline
x^2 + 2ax + a^2 \text{ square} \\
x + a \\
\hline
x^3 + 2ax^2 + a^2x \\
\quad + ax^2 + 2a^2x + a^3 \\
\hline
x^3 + 3ax^2 + 3a^2x + a^3 \text{ cube.}
\end{array}$$

&c.

Examples for Practice.

1. Required the cube, or third power, of $2a^2$.
2. Required the biquadrate, or 4th power, of $2a^2x$.
3. Required the cube, or third power, of $-12x^2y^3$.
4. Required the biquadrate, or 4th power, of $-\frac{3a^2x}{5b^2}$.

5. Required the 5th power of $a+x$; and the 6th power of $a-y$.

RULE II.

A binomial or residual quantity, may also be readily raised to any power whatever, as follows:

1. Find the terms without the coefficients, by observing that the index of the first, or leading quantity, begins with that of the given power, and decreases continually by 1, in every term to the last; and that in the following quantity, the indices of the terms are 1, 2, 3, 4, &c.

2. To find the coefficients, observe that those of the first and last terms are always 1; and that the coefficient of the second term is the index of the power: and for the rest, if the coefficient of any term be multiplied by the index of the leading quantity in it, and the product be divided by the number of terms to that place, it will give the coefficient of the term next following.

Note. The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be +; but if the second term be −, all the odd terms will be +, and the even terms −; or, which is the same thing, the terms will be + and − alternately (*u*).

(*u*) The rule here given, which is the same, in the case of integral powers, as the binomial theorem of Newton, may be expressed in general terms, as follows:

$$(a+b)^m = a^m + ma^{m-1}b + m.\frac{m-1}{2}a^{m-2}b^2 + m.\frac{m-1}{2}.\frac{m-2}{3}a^{m-3}b^3, \&c.$$

$$(a-b)^m = a^m - ma^{m-1}b + m.\frac{m-1}{2}a^{m-2}b^2 - m.\frac{m-1}{2}.\frac{m-2}{3}a^{m-3}b^3, \&c.$$

EXAMPLES.

1. Let $x + a$ be involved, or raised to the 5th power.

Here the terms, without the coefficients, are

$$x^5,\ x^4a,\ x^3a^2,\ x^2a^3,\ xa^4,\ a^5.$$

And the coefficients will be

$$1,\ 5,\ \frac{5 \times 4}{2},\ \frac{10 \times 3}{3},\ \frac{10 \times 2}{4},\ \frac{5 \times 1}{5};$$

$$\text{or } 1,\ 5,\ 10,\ 10,\ 5,\ 1.$$

Whence the entire 5th power of $a + x$ is

$$x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5,$$

$$\text{or } x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5.$$

2. Let $a - x$ be involved, or raised, to the 6th power.

Here the terms, without their coefficients, are

$$a^6,\ a^5x,\ a^4x^2,\ a^3x^3,\ a^2x^4,\ ax^5,\ x^6.$$

And the coefficients are

Which formulæ will, also, equally hold when m is a fraction, as will be more fully explained hereafter.

It may, also, be farther observed, that the sum of the coefficients in every power, is equal to the number 2 raised to that power. Thus, $1 + 1 = 2$, for the first power; $1 + 2 + 1 = 4 = 2^2$, for the square; $1 + 3 + 3 + 1 = 8 = 2^3$, for the cube, or third power; and so on.

The trinomial quantity $a + b + c$ may also be raised to any power in the same way, by considering two of its terms as a single one, and proceeding as before. Thus, $(a + b + c)^3 = a^3 + 3(a + b)a^2 + 3(a + b)^2a + (a + b)^3 = a^3 + b^3 + c^3 - 3abc + 3(ab + ac + bc) \times (a + b + c)$. And $(a + b + c)^m = a^m + m(b + c)a^{m-1} + m.\frac{m-1}{2}(b + c)^{m-2}a + m.\frac{m-1}{2}.\frac{m-2}{3}(b + c)^3a^{m-3}$, &c.; where the powers of $b + c$ may be determined by the general theorem, as usual.

$$1,\ 6,\ \frac{6\times5}{2},\ \frac{15\times4}{3},\ \frac{20\times3}{4},\ \frac{15\times2}{5},\ \frac{6\times1}{6};$$

$$\text{or } 1,\ 6,\ 15,\ 20,\ 15,\ 6,\ 1.$$

Whence the entire 6th power of $a-x$ is

$$a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$$

3. Required the 4th power of $a+x$, and the 5th power of $a-x$.

4. Required the 6th power of $b+x$, and the 7th power of $b-y$.

5. Required the 5th power of $2+x$, and the cube of $a-bx+c$.

EVOLUTION.

(H) Evolution, or the extraction of roots, is the reverse of involution, or the raising of powers; being the method of finding the square root, cube root, &c. of any given quantity.

CASE I.

To find the roots of any powers of a simple quantity.

Rule.

Extract the root of the coefficient for the numeral part, and the root of the letter, or letters, for the literal part; and these, joined together, will be the root required.

And if the quantity proposed be a fraction, its root will be found, by taking the root both of its numerator and denominator.

Note. The square root, the fourth root, or any other even root, of an affirmative quantity, may be either + or −. Thus,

$$\sqrt{a^2}=+a \text{ or } -a, \text{ and } \sqrt[4]{b^4}=+b \text{ or } -b, \text{ \&c.}$$

But the cube root, or any other odd root, of a quantity, will have the same sign as the quantity itself. Thus,

$\sqrt[3]{a^3}=a$; $\sqrt[3]{-a^3}=-a$; and $\sqrt[5]{-a^5}=-a$, &c. (v)

It may here, also, be farther remarked, that any even root of a negative quantity, is unassignable.

Thus, $\sqrt{-a^2}$ cannot be determined, as there is no quantity, either positive or negative, (+ or −), that, when multiplied by itself, will produce $-a^2$.

EXAMPLES.

1. Find the square root of $9x^2$; and the cube root of $8x^3$.

$$\sqrt{9x^2}=\sqrt{9}\times\sqrt{x^2}=3\times x=3x; \text{ and}$$
$$\sqrt[3]{8x^3}=\sqrt[3]{8}\times\sqrt[3]{x^3}=2\times x=2x.$$

2. Find the square root of $\frac{a^2x^2}{4c^2}$; and the cube root of $-\frac{8a^3x^3}{27c^3}$.

$$\sqrt{\frac{a^2x^2}{4c^2}}=\frac{\sqrt{a^2x^2}}{\sqrt{4c^2}}=\frac{ax}{2c}; \text{ and } \sqrt[3]{-\frac{8a^3x^3}{27c^3}}=-\frac{2ax}{3c}.$$

3. It is required to find the square root of $4a^2x^6$.

4. It is required to find the cube root of $-125a^3x^6$.

5. It is required to find the 4th root of $256a^4x^8$.

(v) The reason why $+a$ and $-a$ are each the square root of a^2 is obvious, since $(+a)\times(+a)$, and $(-a)\times(-a)$ are both equal to a^2.

And for the cube root, &c. of a negative quantity, it is plain that

$$(-a)\times(-a)\times(-a)=-a^3; \text{ and } (-a^3)\times(+a^2)=-a^5, \text{ \&c.}$$

6. It is required to find the square root of $\frac{4a^4}{9x^2y^4}$.

7. It is required to find the 5th root of $\frac{32a^5x^{10}}{243}$.

CASE II.

To extract the square root of a compound quantity.

RULE.

Range the terms, of which the quantity is composed, according to the dimensions of some letter in them, and set the root of the first term in the quotient.

Subtract the square of the root, thus found, from the first term, and bring down the two next terms to the remainder, for a dividend.

Divide the dividend, thus found, by double that part of the root already determined, and set the result both in the quotient and divisor.

Multiply the divisor, so increased, by the term of the root last placed in the quotient, and subtract the product from the dividend; and so on, as in common arithmetic.

EXAMPLES.

1. Extract the square root of $x^4-4x^3+6x^2-4x+1$.

$$\begin{array}{rl}
 & x^4-4x^3+6x^2-4x+1\,(x^2-2x+1 \\
 & x^4 \\
\hline
2x^2-2x) & -4x^3+6x^2 \\
 & -4x^3+4x^2 \\
\hline
2x^2-4x+1) & 2x^2-4x+1 \\
 & 2x^2-4x+1 \\
\hline
 & *
\end{array}$$

2. Extract the square root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.

$$4a^4+12a^3x+13a^2x^2+6ax^3+x^4(2a^2+3ax+x^2$$
$$4a^4$$
$$4a^2+3ax)12a^3x+13a^2x^2$$
$$12a^3x+9a^2x^2$$
$$4a^2+6ax+x^2)4a^2x^2+6ax^3+x^4$$
$$4a^2x^2+6ax^3+x^4$$
$$*$$

Note. When the quantity to be extracted has no exact root, the operation may be carried on as far as is thought necessary, or till the regularity of the terms shows the law by which the series would be continued.

EXAMPLE.

1. It is required to extract the square root of $1+x$.

$$1+x(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}\text{ \&c.}$$
$$1$$
$$2+\frac{x}{2})x$$
$$x+\frac{x^2}{4}$$
$$2+x-\frac{x^2}{8})-\frac{x^2}{4}$$
$$-\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64}$$
$$2+x-\frac{x^2}{4}+\frac{x^3}{16})\frac{x^3}{8}-\frac{x^4}{64}$$
$$\frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256}$$
$$2+x-\frac{x^2}{4}+\frac{x^3}{8}\text{\&c.})-\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256}$$
$$-\frac{5x^4}{64}-\frac{5x^5}{128}\text{ \&c.}$$

Here, if the numerators and denominators of the two last terms be each multiplied by 3, which will not alter their values, the root will become

$$1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}+\frac{3.5.7x^5}{2.4.6.8.10}\ \&c.$$

where the law of the series is manifest (*w*).

EXAMPLES FOR PRACTICE.

2. Required the square root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.

3. Required the square root of $x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}$

4. Required the square root of $4x^6-4x^4+13x^2-6x+9$.

5. Required the square root of $x^6+4x^5+10x^4+20x^3+25x^2+24x+16$.

6. It is required to extract the square root of a^2+b.

7. It is required to extract the square root of 2, or of 1 + 1.

(*w*) This manner of deducing the law of a series, or other formula, from the consideration of certain particular cases, is usually called *induction*; and though not a strict method of proof, has been the source of most of the great discoveries that have hitherto been made, both in analysis and physics. But care must be taken, in following this mode of investigation, not to generalise too hastily; as it often happens, that a law, which appears to take place in the first part of a process, is not found to hold good throughout. Thus, in the simple instance of reducing $\frac{531251}{3099750}$ to a decimal, its equivalent value is .17 17 17 49 49 &c. of which the real repeating period is 49, and not 17, as might, at first, be imagined.

CASE III.

To find the roots of powers in general.

RULE.

Find the root of the first term, which place in the quotient; and having subtracted its corresponding power from that term, bring down the second term for a dividend.

Divide this by twice the part of the root above determined, for the square root; by three times the square of it, for the cube root; or by four times the cube of it, for the 4th root, &c. and the quotient will be the next term of the root.

Involve the whole of the root, thus found, to its proper power, which subtract from the given quantity, and divide the first term of the remainder by the same divisor as before; and proceed in this manner, till the whole is finished (x).

(x) As this rule, in high powers, is often found to be very laborious, it may be proper to observe, that the roots of various compound quantities may sometimes be easily discovered, as follows:

Extract the roots of all the simple terms, and connect them together by the sign + or −, as may be judged most suitable for the purpose; then involve the compound root, thus found, to its proper power, and, if it be the same with the given quantity, it is the root required. But if it be found to differ only in some of the signs, change them from + to −, or from − to +, till its power agrees with the given one throughout.

Thus, in the fifth example next following, the root is $2a-3x$, which is the difference of the roots of the first and last terms: and in the third example, the root is $a+b+c$, which is the sum of the roots of the first, fourth, and sixth terms. The same may also be observed of the sixth example, where the root is found from the first and last terms.

EXAMPLES.

1. Required the square root of $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$.

$$\begin{array}{l}
a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4(a^2 - ax + x^2 \\
a^4 \\
\hline
2a^2) - 2a^3x \\
\hline
a^4 - 2a^3x + a^2x^2 \\
\hline
\qquad 2a^2)2a^2x^2 \\
\hline
a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4 \\
\hline
\qquad\qquad *
\end{array}$$

2. Required the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

$$\begin{array}{l}
x^6 + 6x^5 - 40x^3 + 96x - 64(x^2 + 2x - 4 \\
x^6 \\
\hline
3x^4)6x^5 \\
\hline
x^6 + 6x^5 + 12x^4 + 8x^3 \\
\hline
\quad 3x^4) - 12x^4 \\
\hline
x^6 + 6x^5 - 40x^3 + 96x^2 - 64 \\
\hline
\qquad\qquad *
\end{array}$$

3. Required the square root of $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.

4. Required the cube root of $x^6 - 5x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

5. Required the 4th root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.

6. Required the 5th root of $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$.

OF IRRATIONAL QUANTITIES, OR SURDS.

(1) **Irrational** quantities, or surds, are such as have no exact root, being usually expressed by means of the radical sign, or by fractional indices; in which latter case, the numerator shows the power the quantity is to be raised to, and the denominator its root. Thus, $\sqrt{2}$, or $2^{\frac{1}{2}}$, denotes the square root of 2; and $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$, is the square of the cube root of a, &c. (y)

CASE I.

To reduce a rational quantity to the form of a surd.

Rule.

Raise the quantity to a power corresponding with that denoted by the index of the surd; and over this new quantity place the radical sign, or proper index, and it will be of the form required.

Examples.

1. Let 3 be reduced to the form of the square root.

Here $3 \times 3 = 3^2 = 9$; whence $\sqrt{9}$ Ans.

(y) A quantity of this kind, as for instance $\sqrt{2}$, is called an irrational number, or a surd, because no number, either whole or fractional, can be found, which, when multiplied by itself, will produce 2. But its approximate value may be determined to any degree of exactness, by the common rule for extracting the square root, being 1 and certain decimals, which never terminate.

2. Let $2x^2$ be reduced to the form of the cube root.

Here $(2x^2)^3=8x^6$; whence $\sqrt[3]{8x^6}$, or $(8x^6)^{\frac{1}{3}}$ Ans.

3. Let 5 be reduced to the form of the square root.

4. Let $-3x$ be reduced to the form of the cube root.

5. Let $-2a$ be reduced to the form of the fourth root.

6. Let a^2 be reduced to the form of the fifth root.

7. Let $\frac{a}{b\sqrt{a}}$ be reduced to the form of the square root.

Note. In like manner, any rational quantity may be reduced to the form of the surd to which it is joined, and their product be placed under the same index, or radical sign.

EXAMPLES.

1. $2\sqrt{2}=\sqrt{4}\times\sqrt{2}=\sqrt{8}$; and $2\sqrt[3]{2}=\sqrt[3]{8}\times\sqrt[3]{2}=\sqrt[3]{16}$.

2. $3\sqrt{a}=\sqrt{9}\times\sqrt{a}=\sqrt{9a}$; and $\frac{1}{2}\sqrt[3]{4a}=\sqrt[3]{\frac{1}{8}}\times\sqrt[3]{4a}=\sqrt[3]{\frac{4a}{8}}=\sqrt[3]{\frac{a}{2}}$

3. Let $5\sqrt{6}$ be reduced to a simple radical form.

4. Let $\frac{1}{5}\sqrt{5a}$ be reduced to a simple radical form.

5. Let $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^6}}$ be reduced to a simple radical form.

CASE II.

To reduce quantities of different indices, to others that shall have a given index.

RULE.

Divide the indices of the proposed quantities by the given index, and the quotients will be the new indices for those quantities.

Then, over the said quantities, with their new indices, place the given index, and they will be the equivalent quantities required.

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to quantities that shall have the index $\frac{1}{6}$

$\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \times \frac{6}{1} = \frac{6}{2} = 3$, the 1st index;

$\frac{1}{3} \div \frac{1}{6} = \frac{1}{3} \times \frac{6}{1} = \frac{6}{3} = 2$, the 2d index.

Whence $(3^3)^{\frac{1}{6}}$ and $(2^2)^{\frac{1}{6}}$, or $27^{\frac{1}{6}}$ and $4^{\frac{1}{6}}$, are the quantities required.

2. Reduce $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$ to quantities that shall have the common index $\frac{1}{8}$

3. Reduce a^2 and $a^{\frac{1}{2}}$ to quantities that shall have the common index $\frac{1}{4}$

4. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$ to quantities that shall have the common index $\frac{1}{8}$

Note. Surds may also be brought to a common index, by reducing the indices of the quantities to a common denominator, and then involving each of them to the power denoted by its numerator.

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$ to quantities having a common index.

$$3^{\frac{1}{2}}=3^{\frac{3}{6}}=\overline{3^3}|^{\frac{1}{6}}=\overline{27}|^{\frac{1}{6}}$$

$$4^{\frac{1}{3}}=4^{\frac{2}{6}}=\overline{4^2}|^{\frac{1}{6}}=\overline{16}|^{\frac{1}{6}}$$

Whence $\overline{27}|^{\frac{1}{6}}$ and $\overline{16}|^{\frac{1}{6}}$ Ans.

2. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$ to quantities that shall have a common index.

3. Reduce $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ to quantities that shall have a common index.

4. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to quantities that shall have a common index.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to quantities that shall have a common index.

CASE III.

To reduce surds to their most simple forms.

RULE.

Resolve the given number, or quantity, into two factors, one of which shall be the greatest power contained in it, and set the root of this power before the remaining part, with the proper radical sign between them (z).

EXAMPLES.

1. Let $\sqrt{48}$ be reduced to its most simple form.

$$\sqrt{48}=\sqrt{16\times 3}=4\sqrt{3} \quad \text{Ans.}$$

2. Let $\sqrt[3]{108}$ be reduced to its most simple form.

$$\sqrt[3]{108}=\sqrt[3]{27\times 4}=3\sqrt[3]{4} \quad \text{Ans.}$$

(z) When the given surd contains no factor that is an exact power, it is already in its most simple form. Thus, $\sqrt{15}$ cannot be reduced lower, because neither of its factors, 5, or 3, is a square.

Note 1. When any number, or quantity, is prefixed to the surd, that quantity must be multiplied by the root of the factor above mentioned, and the product be then joined to the other part, as before.

EXAMPLES.

1. Let $2\sqrt{32}$ be reduced to its most simple form.

$$2\sqrt{32} = 2\sqrt{16 \times 2} = 8\sqrt{2} \text{ Ans.}$$

2. Let $5\sqrt[3]{24}$ be reduced to its most simple form.

$$5\sqrt[3]{24} = 5\sqrt[3]{8 \times 3} = 10\sqrt[3]{3} \text{ Ans.}$$

Note 2. A fractional surd may also be reduced to a more convenient form, by multiplying both the numerator and denominator by such a number, or quantity, as will make the denominator a complete power, of the kind required; and then joining its root to the other part, as above (*a*).

EXAMPLES.

1. Let $\sqrt{\frac{2}{7}}$ be reduced to its most simple form.

$$\sqrt{\frac{2}{7}} = \sqrt{\frac{14}{49}} = \sqrt{\frac{1}{49} \times 14} = \frac{1}{7}\sqrt{14} \text{ Ans.}$$

2. Let $3\sqrt[3]{\frac{2}{5}}$ be reduced to its most simple form.

$$3\sqrt[3]{\frac{2}{5}} = 3\sqrt[3]{\frac{50}{125}} = 3\sqrt[3]{\frac{1}{125} \times 50} = \frac{3}{5}\sqrt[3]{50} \text{ Ans.}$$

(*a*) The utility of reducing surds to their most simple forms, in order to have the answer in decimals, will be readily perceived from considering the 1st question above given, where it is found that $\sqrt{\frac{2}{7}} = \frac{1}{7}\sqrt{14}$; in which case it is only necessary to extract the square root of the whole number 14, (or to find it in some of the tables that have been calculated for this purpose,) and then divide it by 7; whereas, otherwise, we must have first divided the numerator by the denominator, and then have found the root of the quotient, for the surd part; or else have determined

EXAMPLES FOR PRACTICE.

3. Reduce $\sqrt{125}$ to its most simple form.
4. Reduce $\sqrt{294}$ to its most simple form.
5. Reduce $\sqrt[3]{56}$ to its most simple form.
6. Reduce $\sqrt[3]{192}$ to its most simple form.
7. Reduce $7\sqrt{80}$ to its most simple form.
8. Reduce $9\sqrt[3]{81}$ to its most simple form.
9. Reduce $\frac{3}{121}\sqrt{\frac{5}{6}}$ to its most simple form.
10. Reduce $\frac{4}{7}\sqrt[3]{\frac{3}{16}}$ to its most simple form.
11. Reduce $\sqrt{98a^2x}$ to its most simple form.
12. Reduce $\sqrt{x^3 - a^2x^2}$ to its most simple form.

CASE IV.

To add surd quantities together.

RULE.

When the surds are of the same kind, reduce them to their simplest forms, as in the last case; then, if the surd part be the same in them all, annex it to the sum of the rational parts, and it will give the whole sum required.

But if the quantities have different indices, or the surd part be not the same in them all, they can only be added together by the signs + and −.

EXAMPLES.

1. It is required to find the sum of $\sqrt{27}$ and $\sqrt{48}$.

---

the root both of the numerator and denominator, and then divided the one by the other; which are each of them very troublesome processes when performed by the common rules: and in the next example, for the cube root, the labour would be much greater.

$$\sqrt{27}=\sqrt{9\times 3}=3\sqrt{3}$$
$$\sqrt{48}=\sqrt{16\times 3}=4\sqrt{3}$$
$$\overline{7\sqrt{3}}\text{ the sum.}$$

2. It is required to find the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

$$\sqrt[3]{500}=\sqrt[3]{125\times 4}=5\sqrt[3]{4}$$
$$\sqrt[3]{108}=\sqrt[3]{27\times 4}=3\sqrt[3]{4}$$
$$\overline{8\sqrt[3]{4}}\text{ the sum.}$$

3. It is required to find the sum of $4\sqrt{147}$ and $3\sqrt{75}$.

$$4\sqrt{147}=4\sqrt{49\times 3}=28\sqrt{3}$$
$$3\sqrt{75}=3\sqrt{25\times 3}=15\sqrt{3}$$
$$\overline{43\sqrt{3}}\text{ the sum.}$$

4. It is required to find the sum of $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$.

$$3\sqrt{\frac{2}{5}}=3\sqrt{\frac{10}{25}}=\frac{3}{5}\sqrt{10}$$
$$2\sqrt{\frac{1}{10}}=2\sqrt{\frac{10}{100}}=\frac{2}{10}\sqrt{10}$$
$$\overline{\frac{4}{5}\sqrt{10}}\text{ the sum.}$$

EXAMPLES FOR PRACTICE.

5. It is required to find the sum of $\sqrt{72}$ and $\sqrt{108}$.

6. It is required to find the sum of $\sqrt{180}$ and $\sqrt{405}$.

7. It is required to find the sum of $\sqrt[3]{40}$ and $\sqrt[3]{135}$.

8. It is required to find the sum of $4\sqrt[3]{54}$ and $5\sqrt[3]{128}$.

9. It is required to find the sum of $9\sqrt{243}$ and $10\sqrt{363}$.

10. It is required to find the sum of $3\sqrt{\frac{2}{3}}$ and $7\sqrt{\frac{27}{50}}$.

11. It is required to find the sum of $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$

12. It is required to find the sum of $\frac{1}{2}\sqrt{a^2b}$ and $\frac{1}{3}\sqrt{4bx^4}$.

13. It is required to find the sum of $a\sqrt[3]{a^3b^4}$ and $b\sqrt[3]{8a^6b}$.

CASE V.

To find the difference of surd quantities.

RULE.

When the surds are of the same kind, prepare the quantities as in the last rule; then the difference of the rational parts annexed to the common surd, will give the whole difference required.

But if the quantities have different indices, or the surd part be not the same in both of them, they can only be subtracted by means of the sign $-$.

1. It is required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

$$\sqrt{448}=\sqrt{\overline{64\times 7}}=8\sqrt{7}$$
$$\sqrt{112}=\sqrt{\overline{16\times 7}}=4\sqrt{7}$$

$4\sqrt{7}$ the difference.

2. It is required to find the difference of $\sqrt[3]{192}$ and $\sqrt[3]{24}$.

$$\sqrt[3]{192}=\sqrt[3]{64\times3}=4\sqrt[3]{3}$$
$$\sqrt[3]{24}=\sqrt[3]{8\times3}=2\sqrt[3]{3}$$

$2\sqrt[3]{3}$ the difference.

3. It is required to find the difference of $5\sqrt{20}$ and $3\sqrt{45}$.

$$5\sqrt{20}=5\sqrt{4\times5}=10\sqrt{5}$$
$$3\sqrt{45}=3\sqrt{9\times5}=9\sqrt{5}$$

$\sqrt{5}$ the difference.

4. It is required to find the difference of $\frac{3}{4}\sqrt{\frac{2}{3}}$ and $\frac{2}{5}\sqrt{\frac{1}{6}}$

$$\frac{3}{4}\sqrt{\frac{2}{3}}=\frac{3}{4}\sqrt{\frac{6}{9}}=\frac{3}{12}\sqrt{6}=\frac{1}{4}\sqrt{6}$$
$$\frac{2}{5}\sqrt{\frac{1}{6}}=\frac{2}{5}\sqrt{\frac{6}{36}}=\frac{2}{30}\sqrt{6}=\frac{1}{15}\sqrt{6}$$

$\frac{11}{60}\sqrt{6}$ difference.

EXAMPLES FOR PRACTICE.

5. It is required to find the difference of $2\sqrt{50}$ and $\sqrt{18}$.

6. It is required to find the difference of $\sqrt[3]{320}$ and $\sqrt[3]{40}$.

7. It is required to find the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$

8. It is required to find the difference of $\sqrt[3]{\frac{2}{3}}$ and $\sqrt[3]{\frac{9}{32}}$

9. It is required to find the difference of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$.

10. It is required to find the difference of $8\sqrt[3]{a^3b}$ and $2\sqrt[3]{a^6b}$.

CASE VI.

To multiply surd quantities together.

RULE.

When the surds are of the same kind, find the product of the rational parts, and the product of the surds; and the two joined together, with their common radical sign between them, will give the whole product required; which may be reduced to its most simple form by Case III.

But if the surds are of different kinds, they must be reduced to a common index, and then multiplied together as before.

It is also to be observed, that the product of different powers, or roots, of the same quantity, is found by adding their indices.

EXAMPLES.

1. It is required to find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

$$\begin{array}{l} 3\sqrt{8} \\ 2\sqrt{6} \\ \hline 6\sqrt{48} = 6\sqrt{16 \times 3} = 24\sqrt{3} \text{ Ans.} \end{array}$$

2. It is required to find the product of $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$

$$\begin{array}{l} \frac{1}{2}\sqrt[3]{\frac{2}{3}} \\ \frac{3}{4}\sqrt[3]{\frac{5}{6}} \\ \hline \frac{3}{8}\sqrt[3]{\frac{10}{18}} = \frac{3}{8}\sqrt[3]{\frac{5}{9}} = \frac{3}{8}\sqrt[3]{\frac{15}{27}} = \frac{1}{8}\sqrt[3]{15} \text{ Ans.} \end{array}$$

3. It is required to find the product of $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$.

$$2^{\frac{1}{2}}=2^{\frac{3}{6}}=(2^3)^{\frac{1}{6}}=8^{\frac{1}{6}}$$
$$3^{\frac{1}{3}}=3^{\frac{2}{6}}=(3^2)^{\frac{1}{6}}=9^{\frac{1}{6}}$$
$$72^{\frac{1}{6}} \text{ Ans.}$$

4. It is required to find the product of $5\sqrt{a}$ and $3\sqrt[3]{a}$.

$$5\sqrt{a}=5a^{\frac{1}{2}}=5a^{\frac{3}{6}}$$
$$3\sqrt[3]{a}=3a^{\frac{1}{3}}=3a^{\frac{2}{6}}$$
$$15a^{\frac{5}{6}}=15(a^5)^{\frac{1}{6}} \text{ or } 15\sqrt[6]{a^5} \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

5. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$.
6. Required the product of $7\sqrt[3]{18}$ and $5\sqrt[3]{4}$.
7. Required the product of $\frac{1}{4}\sqrt[3]{6}$ and $\frac{2}{15}\sqrt[3]{9}$.
8. Required the product of $\frac{1}{2}\sqrt[3]{18}$ and $5\sqrt[3]{20}$.
9. Required the product of $2\sqrt{3}$ and $13\frac{1}{2}\sqrt[3]{5}$.
10. Required the product of $2\frac{1}{4}a^{\frac{2}{3}}$ and $120\frac{1}{3}a^{\frac{1}{4}}$
11. Required the product of $4+2\sqrt{2}$ and $2-\sqrt{2}$.
12. Required the continued product of ab, $a(a+\sqrt{c})^{\frac{1}{2}}$ and $b(a-\sqrt{c})^{\frac{1}{2}}$
13. Required the product of $2(a+b)^{\frac{1}{n}}$ and $3(a+b)^{\frac{1}{m}}$

CASE VII.

To divide one surd quantity by another.

RULE.

When the surds are of the same kind, find the quotient of the rational parts, and the quotient of the surds, and the two joined together, with their common radical sign between them, will give the whole quotient required.

But if the surds are of different kinds, they must be reduced to a common index, and then be divided as before.

It is also to be observed, that the quotient of different powers, or roots of the same quantity, is found by subtracting their indices.

EXAMPLES.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

$$\frac{8\sqrt{108}}{2\sqrt{6}} = 4\sqrt{18} = 4\sqrt{9 \times 2} = 12\sqrt{2} \text{ Ans.}$$

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$

$$\frac{8\sqrt[3]{512}}{4\sqrt[3]{2}} = 2\sqrt[3]{256} = 2\sqrt[3]{64 \times 4} = 8\sqrt[3]{4} \text{ Ans.}$$

3. It is required to divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$

$$\frac{\frac{1}{2}\sqrt{5}}{\frac{1}{3}\sqrt{2}} = \frac{3}{2}\sqrt{\frac{5}{2}} = \frac{3}{2}\sqrt{\frac{10}{4}} = \frac{3}{4}\sqrt{10} \text{ Ans.}$$

4. It is required to divide $\sqrt{7}$ by $\sqrt[3]{7}$

$$\frac{\sqrt{7}}{\sqrt[3]{7}} = \frac{7^{\frac{1}{2}}}{7^{\frac{1}{3}}} = \frac{7^{\frac{3}{6}}}{7^{\frac{2}{6}}} = 7^{\frac{3}{6}-\frac{2}{6}} = 7^{\frac{1}{6}} \text{ Ans.}$$

5. It is required to divide $6\sqrt{54}$ by $3\sqrt{2}$.

6. It is required to divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$.

7. It is required to divide $5\frac{3}{4}\sqrt{\frac{1}{135}}$ by $\frac{2}{3}\sqrt{\frac{1}{5}}$

8. It is required to divide $3\frac{5}{7}\sqrt[3]{\frac{2}{3}}$ by $2\frac{2}{5}\sqrt[3]{\frac{3}{4}}$

9. It is required to divide $4\frac{1}{2}\sqrt{a}$ by $2\frac{2}{3}\sqrt[3]{ab}$.

10. It is required to divide $32\frac{2}{5}a^{\frac{1}{2}}$ by $13\frac{3}{4}a^{\frac{1}{3}}$

11. It is required to divide $9\frac{3}{8}a^{\frac{1}{n}}$ by $4\frac{9}{11}a^{\frac{1}{n}}$

12. It is required to divide $\sqrt{20}+\sqrt{12}$ by $\sqrt{5}-\sqrt{3}$.

Note. Since the division of surds is performed by subtracting their indices, it is evident that the denominator of any fraction may be taken into the numerator, or the numerator into the denominator, by changing the sign of its index.

Also, since $\frac{a^m}{a^m}=a^{m-m}=1$, or a^0, it follows, that the expression a^0 is a symbol equivalent to unity, and, consequently, that it may be always replaced, wherever it occurs, by 1 (*b*).

EXAMPLES.

1. Thus, $\frac{1}{a}=\frac{a^{-1}}{1}$, or a^{-1}; and $\frac{1}{a^n}=\frac{a^{-n}}{1}$, or a^{-n}.

2. Also, $\frac{b}{a^2}=\frac{ba^{-2}}{1}$, or ba^{-2}; and $\frac{a^{-n}}{b^{-m}}=\frac{1}{a^n b^{-m}}$, or $\frac{b^m}{a^n}$

3. Let $\frac{1}{a^3}$ be expressed with a negative index.

4. Let $a^{-\frac{1}{2}}$ be expressed with a positive index.

5. Let $\frac{1}{a+x}$ be expressed with a negative index.

6. Let $a(a^2-x^2)^{-\frac{1}{3}}$ be expressed with a positive index.

CASE VIII.

To involve, or raise, surd quantities to any power.

RULE.

Multiply the index of the quantity by the num-

(*b*) This expression, properly speaking, is only the result of an operation, to which we are led by the common convention of representing the powers of quantities by numeral indices.

ber denoting the index of the power to which it is to be raised, and to the result annex the power of the rational parts, and it will give the whole power required (*c*).

Examples.

1. It is required to find the square of $\frac{2}{3}a^{\frac{1}{3}}$

$$\left(\frac{2}{3}a^{\frac{1}{3}}\right)^2 = \frac{4}{9}\left(a^{\frac{1}{3}}\right)^2 = \frac{4}{9}a^{\frac{2}{3}} = \frac{4}{9}\sqrt[3]{a^2} \text{ Ans.}$$

2. It is required to find the cube of $\frac{2}{3}\sqrt{3}$.

$$\left(\frac{2}{3}\sqrt{3}\right)^3 = \frac{8}{27}(\sqrt{3})^3 = \frac{8}{27}\left(3^{\frac{1}{2}}\right)^3 = \frac{8}{27}\left(3^{\frac{3}{2}}\right) = \frac{8}{27}\sqrt{27} =$$
$$\frac{8}{27}\sqrt{9\times 3} = \frac{24}{27}\sqrt{3} = \frac{8}{9}\sqrt{3} \text{ Ans.}$$

3. It is required to find the square of $3\sqrt[3]{3}$.
4. It is required to find the cube of $17\sqrt{21}$.
5. It is required to find the 4th power of $\frac{1}{6}\sqrt{6}$.
6. It is required to find the square of $3+\sqrt{5}$.
7. It is required to find the cube of $\sqrt{x}+\sqrt{y}$.
8. It is required to find the 4th power of $\sqrt{3}-\sqrt{2}$.

CASE IX.

To find the roots of surd quantities.

Rule.

Multiply the index of the given quantity by $\frac{1}{2}$ for

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(*c*) The *n*th power, or root, of a fraction, is equal to the *n*th power, or root, of the numerator, divided by the *n*th power, or root, of the denominator. Thus,

$$\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}, \text{ and } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}; \text{ also, } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{ab}}{b}, \text{ and } \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$$

the square root, by $\frac{1}{3}$ for the cube root, or $\frac{1}{4}$ for the 4th root, &c. and it will give the root of the surd part; to which annex the root of the rational part, and it will give the whole root required (*d*).

Examples.

1. It is required to find the square root of $9\sqrt[3]{3}$.

$$(9\sqrt[3]{3})^{\frac{1}{2}} = 9^{\frac{1}{2}}(3^{\frac{1}{3}})^{\frac{1}{2}} = 9^{\frac{1}{2}} \times 3^{\frac{1}{6}} = 3\sqrt[6]{3} \text{ Ans.}$$

2. It is required to find the cube root of $\frac{1}{8}\sqrt{2}$.

$$(\tfrac{1}{8}\sqrt{2})^{\frac{1}{3}} = (\tfrac{1}{8})^{\frac{1}{3}} \times (2^{\frac{1}{2}})^{\frac{1}{3}} = \tfrac{1}{2}(2^{\frac{1}{6}}) = \tfrac{1}{2}\sqrt[6]{2} \text{ Ans.}$$

3. It is required to find the square root of 10^3.

4. It is required to find the cube root of $\frac{8}{27}a^4$.

5. It is required to find the 4th root of $\frac{16}{81}a^{\frac{2}{3}}$

6. It is required to find the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$

7. It is required to find the square root of $x^2 - 4x\sqrt{a} + 4a$.

8. It is required to find the square root of $a + 2\sqrt{ab} + b$.

(*d*) The *n*th root of the *m*th power of *a*, or the *m*th power of the *n*th root of *a*, is $a^{\frac{m}{n}}$.

Also, the *n*th root of the *m*th root of *a*, or the *m*th root of the *n*th root of *a*, is $a^{\frac{1}{mn}}$.

From which last expression, it readily appears, that the square root of the square root of *a* is the 4th root of *a*; and that the cube root of the square root of *a*, or the square root of the cube root of *a*, is the 6th root of *a*; and so on for any other numerical root of this kind.

CASE X.

To change a binomial, or a residual surd, into a general surd.

RULE.

Involve the given binomial, or residual, to a power equivalent to that denoted by the surd; then set the radical sign of the same root over it, and it will be the general surd required.

EXAMPLES.

1. It is required to reduce $2+\sqrt{3}$ to a general surd.

Here $(2+\sqrt{3})^2=4+3+4\sqrt{3}=7+4\sqrt{3}$; therefore $2+\sqrt{3}=\sqrt{7+4\sqrt{3}}$.

2. It is required to reduce $\sqrt{2}+\sqrt{3}$ to a general surd.

Here $(\sqrt{2}+\sqrt{3})^2=2+3+2\sqrt{6}=5+2\sqrt{6}$; therefore $\sqrt{2}+\sqrt{3}=\sqrt{5+2\sqrt{6}}$.

3. It is required to reduce $\sqrt[3]{2}+\sqrt[3]{4}$ to a general surd.

Here $(\sqrt[3]{2}+\sqrt[3]{4})^3=6+6\sqrt[3]{2}+6\sqrt[3]{4}$; therefore $\sqrt[3]{2}+\sqrt[3]{4}=\sqrt[3]{6+6\sqrt[3]{2}+6\sqrt[3]{4}}$.

4. It is required to reduce $3-\sqrt{5}$ to a general surd.

5. It is required to reduce $\sqrt{2}-2\sqrt{6}$ to a general surd.

6. It is required to reduce $4-\sqrt{7}$ to a general surd.

7. It is required to reduce $2\sqrt[3]{3}-3\sqrt[3]{9}$ to a general surd.

CASE XI.

To extract the square root of a binomial, or residual surd.

RULE.

Substitute the numbers, or parts of which the given surd is composed, in the place of the letters, in one of the two following formulæ, according as it is a binomial or a residual, and it will give the root required.

$$\sqrt{a+\sqrt{b}}=\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}+\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}$$

$$\sqrt{a-\sqrt{b}}=\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}-\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}$$

Where it is to be observed, that if both a and $\sqrt{a^2-b}$, in these formulæ, be rational quantities, the root will consist either of two surds, or of a rational part and a surd, which are the only cases of the rule that are useful (*e*).

(*e*) If the second part of the binomial, in this case, be an imaginary surd, the same theorems will still hold, by only changing $-b$ into $+b$, as below:

$$\sqrt{a+\sqrt{-b}}=\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2+b}}+\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2+b}}$$

$$\sqrt{a-\sqrt{-b}}=\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2+b}}-\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2+b}}$$

Thus, if it were required to find the square root of $1 \pm \sqrt{-48}$, we shall have

$$\sqrt{1+\sqrt{-48}}=\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+48}}+\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1+48}}=\sqrt{\tfrac{1}{2}+\tfrac{7}{2}}+\sqrt{\tfrac{1}{2}-\tfrac{7}{2}}$$
$$=\sqrt{4}+\sqrt{-3}=2+\sqrt{-3};\text{ and }\sqrt{1-\sqrt{-48}}=1-\sqrt{-3}.$$

The cube root of any binomial surd, of the form

$$\sqrt[3]{a+\sqrt{\pm b}},\text{ or }\sqrt{a-\sqrt{\pm b}},$$

may also be determined, if $a^2 \pm b$ be a perfect integral cube, and

EXAMPLES.

1. It is required to find the square root of $11+\sqrt{72}$.

Here

$$\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}=\sqrt{\tfrac{11}{2}+\tfrac{1}{2}\sqrt{121-72}}=\sqrt{\tfrac{11}{2}+\tfrac{7}{2}}=3;$$

and

$$\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}=\sqrt{\tfrac{11}{2}-\tfrac{1}{2}\sqrt{121-72}}=\sqrt{\tfrac{11}{2}-\tfrac{7}{2}}=\sqrt{2}:$$

whence $\sqrt{11+\sqrt{72}}$, or $\sqrt{11+6\sqrt{2}}=3+\sqrt{2}$ Ans.

2. It is required to find the square root of $3-2\sqrt{2}$.

Here

$$\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}=\sqrt{\tfrac{3}{2}+\tfrac{1}{2}\sqrt{9-8}}=\sqrt{\tfrac{3}{2}+\tfrac{1}{2}}=\sqrt{2};\text{ and}$$

$$-\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}=-\sqrt{\tfrac{3}{2}-\tfrac{1}{2}\sqrt{9-8}}=-\sqrt{\tfrac{3}{2}-\tfrac{1}{2}}=-1:$$

whence $\sqrt{3-2\sqrt{2}}=\sqrt{2}-1$ Ans.

3. It is required to find the square root of $6-2\sqrt{5}$.

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some whole number, or the half of some whole number, can be found, that, when substituted for n, will make

$$4n^3-3(\sqrt[3]{a^2\mp b})n=a.$$

In which case the roots of the two expressions are as follows:

$$\sqrt[3]{a+\sqrt{\pm b}}=n+\sqrt{n^2-\sqrt[3]{a^2\mp b}}$$

$$\sqrt[3]{a-\sqrt{\pm b}}=n-\sqrt{n^2-\sqrt[3]{a^2\mp b}}$$

Thus, if it were required to find $\sqrt[3]{10\pm\sqrt{108}}$, we shall have $4n^3-3(\sqrt[3]{100-108})n$, or $4n^3+6n=10$, where it readily appears that $n=1$; whence $\sqrt[3]{10+\sqrt{108}}=1+\sqrt{1+2}=1+\sqrt{3}$; and $\sqrt[3]{10-\sqrt{108}}=1-\sqrt{3}$.

And if it were required to find $\sqrt[3]{2\pm\sqrt{-121}}$, we shall have $4n^3-3(\sqrt[3]{a^2+b})n$, or $4n^3-15n=2$; where it is evident that $n=2$; whence $\sqrt[3]{2+\sqrt{-121}}=2+\sqrt{4-5}=2+\sqrt{-1}$; and $\sqrt[3]{2-\sqrt{-121}}=2-\sqrt{-1}$.

$$\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}=\sqrt{\tfrac{6}{2}+\tfrac{1}{2}\sqrt{36-20}}=\sqrt{\tfrac{6}{2}+\tfrac{4}{2}}=\sqrt{5};$$

and

$$-\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}=-\sqrt{\tfrac{6}{2}-\tfrac{1}{2}\sqrt{36-20}}=-\sqrt{\tfrac{6}{2}-\tfrac{4}{2}}=-1:$$

whence $\sqrt{6-2\sqrt{5}}=\sqrt{5}-1$ Ans.

4. It is required to find the square root of $2+\sqrt{3}$.

5. It is required to find the square root of $65-20\sqrt{10}$.

6. It is required to find the square root of $17+2\sqrt{70}$.

7. It is required to find the square root of $a^2+2x\sqrt{a^2-x^2}$.

8. It is required to find the square root of $6+2\sqrt{2}-2\sqrt{3}-2\sqrt{6}$.

CASE XII.

To find such a multiplier, or multipliers, as will make any binomial surd rational.

RULE.

1. When one or both the terms are any even roots; multiply the given binomial, or residual, by the same quantity, with its sign changed; and repeat the operation as long as there are surds, when the last result will be rational.

In like manner, a trinomial surd may also be rendered rational, by changing the sign of one of its terms for the multiplier; and a quadrinomial surd by changing the signs of two of the terms, &c.

2. When the terms of the binomial surd are odd roots, the rule becomes more complicated; but for the sum, or difference, of two cube roots, which is

the most useful case, the multiplier will be a trinomial surd consisting of the squares of the two given terms, and their product, with its sign changed(*f*).

EXAMPLES.

1. To find a multiplier that shall render $5+\sqrt{3}$ rational.

Given surd $5+\sqrt{3}$
Multiplier $5-\sqrt{3}$

Product $25-3=22$

2. To find a multiplier that shall make $\sqrt{5}+\sqrt{3}$ rational.

Given surd $\sqrt{5}+\sqrt{3}$
Multiplier $\sqrt{5}-\sqrt{3}$

Product $5-3=2$

3. To find multipliers that shall make $\sqrt[4]{5}+\sqrt[4]{3}$ rational.

Given surd $\sqrt[4]{5}+\sqrt[4]{3}$
1st multiplier $\sqrt[4]{5}-\sqrt[4]{3}$

1st product $\sqrt{5}-\sqrt{3}$
2d multiplier $\sqrt{5}+\sqrt{3}$

2d product $5-3=2$

(*f*) If a multiplier be required, that shall render any binomial surd, whether it consist of even or odd roots, rational, it may be found by substituting the given numbers, or letters, of which it is formed, in the places of their equals, in the following general formula:

Binomial $\sqrt[n]{a}\pm\sqrt[n]{b}$
Multiplier $\sqrt[n]{a^{n-1}}\mp\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}\mp\sqrt[n]{a^{n-4}b^3}+$ &c.

Where the upper sign of the multiplier must be taken with the

4. To find multipliers that shall make $\sqrt{5}+\sqrt{3}-\sqrt{2}$ rational.

$$\begin{array}{lr}
\text{Given surd} & \sqrt{5}+\sqrt{3}-\sqrt{2} \\
\text{1st multiplier} & \sqrt{5}+\sqrt{3}+\sqrt{2} \\
\hline
 & 5+\sqrt{15}-\sqrt{10} \\
 & +\sqrt{15}+3-\sqrt{6} \\
 & +\sqrt{10}+\sqrt{6}-2 \\
\hline
\text{1st product} & 6+2\sqrt{15} \\
\text{2d multiplier} & -6+2\sqrt{15} \\
\hline
 & -36-12\sqrt{15} \\
 & +12\sqrt{15}+60 \\
\hline
\text{2d product} & 60-36=24
\end{array}$$

5. To find a multiplier that shall make $\sqrt[3]{7}+\sqrt[3]{3}$ rational.

$$\begin{array}{lr}
\text{Given surd} & \sqrt[3]{7}+\sqrt[3]{3} \\
\text{Multiplier} & \sqrt[3]{7^2}-\sqrt[3]{7\times 3}+\sqrt[3]{3^2} \\
\hline
 & 7+\sqrt[3]{3\times 7^2} \\
 & -\sqrt[3]{3\times 7^2}-\sqrt[3]{7\times 3^2} \\
 & +\sqrt[3]{7\times 3^2}+3 \\
\hline
\text{Product} & 7+3=10
\end{array}$$

6. To find a multiplier that shall make $\sqrt{5}-\sqrt{x}$ rational.

7. To find a multiplier that shall make $\sqrt{a}+\sqrt{b}$ rational.

8. To find multipliers that shall make $a+\sqrt[4]{b}$ rational.

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upper sign of the binomial, and the lower with the lower; and the series continued to n terms.

9. To find multipliers that shall make $\sqrt[3]{a}+\sqrt[3]{b}$ rational.

10. To find multipliers that shall make $\sqrt{a}+\sqrt{b}+\sqrt{c}$ rational.

11. To find multipliers that shall make $\sqrt{a}+\sqrt{b}-\sqrt{c}+\sqrt{d}$ rational.

12. It is required to find a multiplier that shall render $1-\sqrt[3]{2a}$ rational.

13. It is required to find a multiplier that shall render $2\sqrt[3]{3}-\frac{1}{2}\sqrt[3]{2}$ rational.

CASE XIII.

To reduce a fraction, whose denominator is either a simple or a compound surd, to another that shall have a rational denominator.

RULE.

1. When any simple fraction is of the form $\frac{b}{\sqrt{a}}$, multiply each of its terms by $\sqrt{a}$, and the resulting fraction will be $\frac{b\sqrt{a}}{a}$; or when it is of the form $\frac{b}{\sqrt[3]{a}}$, multiply both the terms by $\sqrt[3]{a^2}$, and the result will be $\frac{b\sqrt[3]{a^2}}{a}$; and for the general form $\frac{b}{\sqrt[n]{a}}$, multiply by $\sqrt[n]{a^{n-1}}$, and the result will be $\frac{b\sqrt[n]{a^{n-1}}}{a}$.

2. For a compound surd, find such a multiplier, by one of the rules of the last case, as will make the denominator rational; and multiply both the numerator and denominator by it.

EXAMPLES.

1. Reduce the fractions $\frac{2}{\sqrt{3}}$, and $\frac{3}{4\sqrt[3]{5}}$, to others that shall have rational denominators.

Here $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$; and $\frac{3}{\frac{1}{2}\sqrt[4]{5}} = \frac{3}{\frac{1}{2}\sqrt[4]{5}} \times \frac{\sqrt[4]{5^3}}{\sqrt[4]{5^3}} =$

$$\frac{3\sqrt[4]{5^3}}{\frac{1}{2} \times 5} = \frac{6\sqrt[4]{5^3}}{5} = \frac{6}{5}\sqrt[4]{125} \text{ Ans.}$$

2. Reduce $\frac{3}{\sqrt{5}-\sqrt{2}}$ to a fraction, whose denominator shall be rational.

Here $\frac{3}{\sqrt{5}-\sqrt{2}} \times \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}} = \frac{3\sqrt{5}+3\sqrt{2}}{5-2} = \frac{3\sqrt{5}+3\sqrt{2}}{3} =$

$$\frac{\sqrt{5}+\sqrt{2}}{1} = \sqrt{5}+\sqrt{2} \text{ Ans.}$$

3. Reduce $\frac{\sqrt{2}}{3-\sqrt{2}}$ to a fraction, whose denominator shall be rational.

Here $\frac{\sqrt{2}}{3-\sqrt{2}} = \frac{\sqrt{2} \times (3+\sqrt{2})}{(3-\sqrt{2}) \times (3+\sqrt{2})} = \frac{3\sqrt{2}+2}{9-2} = \frac{2+3\sqrt{2}}{7} =$

$$\frac{2}{7}+\frac{3}{7}\sqrt{2} \text{ Ans.}$$

4. Reduce $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$ to a fraction, that shall have a rational denominator.

5. Reduce $\frac{a-\sqrt{b}}{a+\sqrt{b}}$ to a fraction, the denominator of which shall be rational.

6. Reduce $\frac{10}{\sqrt[3]{7}-\sqrt[3]{5}}$ to a fraction, that shall have a rational denominator.

7. Reduce $\frac{2}{\sqrt[4]{5}+\sqrt[4]{3}}$ to a fraction, that shall have a rational denominator.

8. Reduce $\frac{8}{\sqrt{3}+\sqrt{2}+1}$ to a fraction, that shall have a rational denominator.

9. Reduce $\frac{1}{\sqrt{5}+\sqrt{7}+\sqrt{11}}$ to a fraction, that shall have a rational denominator.

OF IMAGINARY QUANTITIES.

(K) IMAGINARY quantities are such expressions as arise from extracting any even root of a negative quantity, as $\sqrt{-a}$, $\sqrt[4]{-a}$, $a+\sqrt{-b}$, &c.; and, though their values are unassignable in numbers, they are yet of great use in some of the higher branches of analysis, as well as in pointing out the conditions under which certain questions are possible or impossible.

Thus, if it should be proposed to divide the number 12, into two such parts that their product may be 40, we shall have, for the two factors required, the imaginary quantities $6+\sqrt{-4}$, and $6-\sqrt{-4}$; which, being of no determinate, or conceivable value, denote that the question, under these conditions, is impossible: and the same will be found to take place in many other cases, where a problem is stated in general terms, or without any regard to the congruity or incongruity of the answer.

This being premised, it is only necessary farther to observe, that the method of adding and subtracting imaginary radicals, is the same as for real quantities; there being no other difference in the common operations with these symbols, than what arises in multiplication, division, and involution, which are performed as follows (g):

(g) It may here be observed, that though quantities, or expressions, of this kind, when taken singly, are of no assignable value; yet, when incorporated by means of the operations of addition, subtraction, multiplication, &c. they frequently pro-

RULE.

1. When one imaginary radical is to be multiplied by another, the result, whether they be both positive or both negative, is equal to *minus* the square root of their product, taking them as real quantities. Thus,

$$(+\sqrt{-a}) \times (+\sqrt{-b}), \text{ or}$$
$$(-\sqrt{-a}) \times (-\sqrt{-b}) = -\sqrt{ab};$$

and, consequently,

$$(+\sqrt{-a}) \times (+\sqrt{-a}), \text{ or}$$
$$(-\sqrt{-a}) \times (-\sqrt{-a}) = -a.$$

And if one of the imaginary radicals be positive and the other negative, the result arising from their multiplication will be *plus* the square root of their product, taking them as before. Thus,

$$(+\sqrt{-a}) \times (-\sqrt{-b}) = +\sqrt{ab}, \text{ or}$$
$$(+\sqrt{-a}) \times (-\sqrt{-a}) = +a.$$

2. When one imaginary radical is to be divided by another, the result, whether they be both positive or both negative, will be equal to *plus* the square root of their quotient, taking them as real quantities. Thus,

$$\frac{+\sqrt{-a}}{+\sqrt{-b}}, \text{ or } \frac{-\sqrt{-a}}{-\sqrt{-b}} = +\sqrt{\frac{a}{b}}; \text{ and } \frac{+\sqrt{-a}}{+\sqrt{-a}}, \text{ or } \frac{-\sqrt{-a}}{-\sqrt{-a}} = 1.$$

And if one of the imaginary radicals be positive and the other negative, the result arising from division will be *minus* the square root of their quotient, taking them as before. Thus,

$$\frac{+\sqrt{-a}}{-\sqrt{-b}}, \text{ or } \frac{-\sqrt{-a}}{+\sqrt{-b}} = -\sqrt{\frac{a}{b}}; \text{ and } \frac{+\sqrt{-a}}{-\sqrt{-a}}, \text{ or } \frac{-\sqrt{-a}}{+\sqrt{-a}} = -1.$$

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duce real results; as will appear from the examples following the rules given in the text, as well as in many other parts of the present work.

3. If an imaginary radical is to be divided by a real radical, or a real radical by an imaginary one, the result will be equal to the square root of *minus* their quotient. Thus,

$$\frac{\sqrt{-a}}{\sqrt{b}}, \text{ or } \frac{\sqrt{a}}{\sqrt{-b}} = +\sqrt{-\frac{a}{b}}; \text{ and } \frac{\sqrt{-a}}{\sqrt{a}}, \text{ or } \frac{\sqrt{a}}{\sqrt{-a}} = +\sqrt{-1}.$$

4. The several powers of $\sqrt{-1}$, to which form the radical part of every simple imaginary surd can always be readily reduced, are as below:

$$\begin{array}{l|l}
\sqrt{-1} \;\; = +\sqrt{-1} & (\sqrt{-1})^5 = +\sqrt{-1} \\
(\sqrt{-1})^2 = -1 & (\sqrt{-1})^6 = -1 \\
(\sqrt{-1})^3 = -\sqrt{-1} & (\sqrt{-1})^7 = -\sqrt{-1} \\
(\sqrt{-1})^4 = +1 & (\sqrt{-1})^8 = +1 \\
\quad \&c. & \quad \&c.
\end{array}$$

Or any power of the same expression may be derived, universally, from the two following formulæ:

$$(\sqrt{-1})^{2n} = \mp 1; \; (\sqrt{-1})^{2n+1} = \mp \sqrt{-1}.$$

Where the upper sign must be taken when n is an odd number, and the under sign when it is an even number.

EXAMPLES.

1. It is required to multiply $1+\sqrt{-1}$ by $1+\sqrt{-1}$, or to find the square of $1+\sqrt{-1}$.

$$\begin{array}{l}
1+\sqrt{-1} \\
1+\sqrt{-1} \\
\hline
1+\sqrt{-1} \\
\quad +\sqrt{-1}-1 \\
\hline
1+2\sqrt{-1}-1 = 2\sqrt{-1} \;\; \text{Ans.}
\end{array}$$

2. It is required to find the product arising from multiplying $1+\sqrt{-1}$ by $1-\sqrt{-1}$.

$$\begin{array}{l} 1+\sqrt{-1} \\ 1-\sqrt{-1} \\ \hline 1+\sqrt{-1} \\ \quad -\sqrt{-1}+1 \\ \hline 1+1=2 \text{ Ans.} \end{array}$$

3. It is required to multiply $a-\sqrt{-b}$ by $a-\sqrt{-b}$, or to find the square of $a-\sqrt{-b}$.

$$\begin{array}{l} a-\sqrt{-b} \\ a-\sqrt{-b} \\ \hline a^2-a\sqrt{-b} \\ \quad -a\sqrt{-b}-b \\ \hline a^2-2a\sqrt{-b}-b \text{ Ans.} \end{array}$$

4. It is required to find the quotient of $1+\sqrt{-1}$ divided by $1-\sqrt{-1}$.

Here $\frac{1+\sqrt{-1}}{1-\sqrt{-1}} = \frac{(1+\sqrt{-1})^2}{2} = \frac{2\sqrt{-1}}{2} = \sqrt{-1}$. Ans.

5. It is required to find the square, or second power, of $a+b\sqrt{-1}$. Ans. $a^2-b^2+2ab\sqrt{-1}$.

6. It is required to find the cube, or third power, of $a-b\sqrt{-1}$. Ans. $a^3+b^3-3ab(b+a\sqrt{-1})$.

7. It is required to find the quotient arising from dividing $3+2\sqrt{-1}$ by $3-2\sqrt{-1}$.

Ans. $\frac{1}{13}(5+12\sqrt{-1})$.

8. It is required to find the square of $\sqrt{a+b\sqrt{-1}}+\sqrt{a-b\sqrt{-1}}$. Ans. $2a+2\sqrt{a^2+b^2}$.

9. It is required to find the cube of $\sqrt[3]{a+b\sqrt{-1}}+\sqrt[3]{a-b\sqrt{-1}}$.

Ans. $2a+3\sqrt[3]{a^2+b^2}(\sqrt[3]{a+b\sqrt{-1}}+\sqrt[3]{a-b\sqrt{-1}})$.

10. It is required to find the square of $\sqrt[4]{a+b\sqrt{-1}}$ $+\sqrt[4]{a-b\sqrt{-1}}$.

Ans. $\sqrt{2a+2\sqrt{a^2+b^2}}+2\sqrt[4]{a^2+b^2}$.

OF ARITHMETICAL PROPORTION AND PROGRESSION.

(L) **Arithmetical proportion**, is the relation which two quantities, of the same kind, have to each other with respect to their difference.

Hence, agreeably to this definition, three quantities are said to be in arithmetical proportion, when the difference of the first and second is equal to the difference of the second and third.

Thus, 2, 4, 6, and a, $a+b$, $a+2b$, are quantities in arithmetical proportion.

And, for the same reason, four quantities are in arithmetical proportion, when the difference of the first and second is equal to the difference of the third and fourth.

Thus, 3, 7, 12, 16, and a, $a+b$, c, $c+b$, are quantities in arithmetical proportion.

Arithmetical progression is when a series of quantities increase or decrease by the same common difference.

Thus, 1, 3, 5, 7, 9, &c. and a, $a+d$, $a+2d$, $a+3d$, &c. are increasing series in arithmetical progression, the common differences of which are 2 and d.

And 15, 12, 9, 6, &c. and a, $a-d$, $a-2d$, $a-3d$, &c. are decreasing series in arithmetical progression, whose common differences are 3 and d.

The most useful properties of arithmetical proportion and progression are contained in the following theorems:

1. If four quantities are in arithmetical proportion, the sum of the two extremes will be equal to the sum of the two means.

Thus, if the proportionals be 2, 5, 7, 10, or a, b, c, d; then will $2+10=5+7$, and $a+d=b+c$.

2. And if three quantities be in arithmetical proportion, the sum of the two extremes will be double the mean.

Thus, if the proportionals be 3, 6, 9, or a, b, c, then will $3+9=2\times6=12$, and $a+c=2b$.

3. Hence an arithmetical mean between any two quantities is equal to half the sum of those quantities.

Thus, an arithmetical mean between 2 and 4 is $=\frac{2+4}{2}=3$; and an arithmetical mean between a and b is $=\frac{a+b}{2}$ (h).

(h) If two, or more, arithmetical means between any two quantities be required, they may be expressed in algebraic terms as below:

Thus, $\frac{2a+b}{3}$ and $\frac{a+2b}{3}=$ two arithmetical means between a and b, a being the less extreme and b the greater.

And $\frac{na+b}{n+1}$, $\frac{(n-1)a+2b}{n+1}$, $\frac{(n-2)a+3b}{n+1}$, &c. to $\frac{a+nb}{n+1}=$ any number (n) of arithmetical means between a and b.

4. In any continued arithmetical progression, the sum of the two extremes is equal to the sum of any two terms that are equally distant from them, or to double the middle term, when the number of terms is odd.

Thus, if the series be 2, 4, 6, 8, 10, &c. then will $2+10=4+8=2\times6=12$.

And, if the series be a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c. then will $a+(a+4d)=a+d+(a+3d)=2\times(a+2d)$.

5. The last term of any increasing arithmetical series is equal to the first term *plus* the product of the common difference by the number of terms less one; and if the series be decreasing, it will be equal to the first term *minus* that product.

Thus, the nth term of the series a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c. is $a+(n-1)d$.

And the nth term of the series a, $a-d$, $a-2d$, $a-3d$, $a-4d$, &c. is $a-(n-1)d$.

6. The sum of any series of quantities in arithmetical progression is equal to the sum of the two extremes multiplied by half the number of terms.

Thus, the sum of 2, 4, 6, 8, 10, 12, is $=(2+12)\times\frac{6}{2}=14\times3=42$.

And the sum of $a+(a+d)+(a+2d)+(a+3d)$, &c. . . . $+l$, is $=(a+l)\times\frac{n}{2}$, where l is the last term, and n the number of terms.

Or, the sum of any increasing arithmetical series may be found, by adding the product of the common difference by the number of terms less one, to twice the first term, and then multiplying the result by half the number of terms.

And, if the series be decreasing, its sum will be found by subtracting the above product from twice the first term, and then multiplying the result by half the number of terms, as before.

Thus, the sum of $a+(a+d)+(a+2d)+(a+3d)+(a+4d)$, &c. to n terms, is $=\overline{2a+(n-1)d}\times\frac{n}{2}$

And the sum of $a+(a-d)+(a-2d)+(a-3d)+(a-4d)$, &c. to n terms, is $=\overline{2a-(n-1)d}\times\frac{n}{2}$ (*i*).

EXAMPLES.

1. The first term of an increasing arithmetical series is 3, the common difference 2, and the num- of terms 20; required the sum of the series.

First, $3+2(20-1)=3+2\times19=3+38=41$, the last term.

And $(3+41)\times\frac{20}{2}=44\times\frac{20}{2}=44\times10=440$, the sum required.

(*i*) The sum of any number of terms (n) of the series of natural numbers 1, 2, 3, 4, 5, 6, 7, &c. is $=\frac{n(n+1)}{2}$

Thus, $1+2+3+4+5$, &c. continued to 100 terms, is $=\frac{100\times101}{2}=50\times101=5050$.

Also, the sum of any number of terms (n) of the series of odd numbers 1, 3, 5, 7, 9, 11, &c. is $=n^2$.

And if any three of the quantities a, d, n, s, be given, the fourth may be found from the equation

$$s=\{2a\pm(n-1)d\}\times\frac{n}{2}, \text{ or } (a+l)\times\frac{n}{2}$$

Where the upper sign + is to be used when the series is increasing, and the lower sign − when it is decreasing; also the last term $l=a\pm(n-1)d$, as above.

Or, $(2 \times 3 + \overline{20 - 1} \times 2) \times \frac{20}{2} = (6 + 19 \times 2) \times 10 = (6 + 38) \times 10 = 44 \times 10 = 440$, as before.

2. The first term of a decreasing arithmetical series is 100, the common difference 3, and the number of terms 34; required the sum of the series.

First, $100 - 3(34 - 1) = 100 - 3 \times 33 = 100 - 99 = 1$, the last term.

And $(100 + 1) \times \frac{34}{2} = 101 \times \frac{34}{2} = 101 \times 17 = 1717$, the sum required.

Or, $\{2 \times 100 - (34 - 1) \times 3\} \times \frac{34}{2} = (200 - 33 \times 3) \times 17 = (200 - 99) \times 17 = 101 \times 17 = 1717$, as before.

3. Required the sum of the natural numbers 1, 2, 3, 4, 5, 6, &c. continued to 1000 terms.

Ans. 500500.

4. Required the sum of the odd numbers 1, 3, 5, 7, 9, &c. continued to 101 terms.

Ans. 10201.

5. How many strokes do the clocks of Venice, which go on to 24 o'clock, strike in the compass of a day? Ans. 300.

6. The first term of a decreasing arithmetical series is 10, the common difference $\frac{1}{3}$, and the number of terms 21; required the sum of the series.

Ans. 140.

7. One hundred stones being placed on the ground, in a straight line, at the distance of a yard from each other; how far will a person travel, who

shall bring them, one by one, to a basket, placed at the distance of a yard from the first stone?

Ans. 5 miles and 1300 yards.

OF GEOMETRICAL PROPORTION AND PROGRESSION.

(M) GEOMETRICAL PROPORTION is that relation of two quantities of the same kind, which arises from considering what part, or parts, the one is of the other, or how often it is contained in it.

When four quantities are compared together, the first and third are called the antecedents, and the second and fourth the consequents.

Ratio is the quotient, which arises from dividing the antecedent by the consequent, or the consequent by the antecedent; observing always to follow the same method.

Hence, three quantities are said to be in geometrical proportion, when the first is the same part, or multiple, of the second, as the second is of the third.

Thus, 3, 6, 12, and a, ar, ar^2, are quantities in geometrical proportion.

And four quantities are said to be in geometrical proportion, when the first is the same part, or multiple, of the second, as the third is of the fourth.

Thus, 2, 8, 3, 12, and a, ar, b, br, are geometrical proportionals.

Direct proportion, is when the same relation subsists between the first of four terms and the second, as between the third and fourth.

Thus, 3, 6, 5, 10, and a, ar, b, br, are in direct proportion.

Inverse, or reciprocal proportion, is when the first and second of four quantities are directly proportional to the reciprocals of the third and fourth.

Thus, 2, 6, 9, 3, and a, ar, br, b, are inversely proportional; because 2, 6, $\frac{1}{9}$, $\frac{1}{3}$, and a, ar, $\frac{1}{br}$, $\frac{1}{b}$, are directly proportional.

Geometrical progression is when a series of quantities have the same constant ratio; or which increase, or decrease, by a common multiplier, or divisor.

Thus, 2, 4, 8, 16, 32, 64, &c. and a, ar, ar^2, ar^3, ar^4, &c. are series in geometrical progression.

The most useful properties of geometrical proportion and progression are contained in the following theorems:

1. If three quantities be in geometrical proportion, the product of the two extremes will be equal to the square of the mean.

Thus, if the proportionals be 2, 4, 8, or a, b, c, then will $2 \times 8 = 4^2$ and $a \times c = b^2$.

2. Hence, a geometrical mean proportional, between any two quantities, is equal to the square root of their product.

Thus, a geometric mean between 4 and 9 is $= \sqrt{36} = 6$.

And a geometric mean between a and b is $= \sqrt{ab}$ (k).

(k) If two, or more, geometrical means between any two quan-

3. If four quantities be in geometrical proportion, the product of the two extremes will be equal to that of the means.

Thus, if the proportionals be 2, 4, 6, 12, or a, b, c, d; then will $2 \times 12 = 4 \times 6$, and $a \times d = b \times c$.

4. Hence, the product of the means of four proportional quantities, divided by either of the extremes, will give the other extreme; and the product of the extremes, divided by either of the means, will give the other mean.

Thus, if the proportionals be 3, 9, 5, 15, or a, b, c, d; then will $\frac{9 \times 5}{3} = 15$, and $\frac{3 \times 15}{5} = 9$: also, $\frac{b \times c}{a} = d$, and $\frac{a \times d}{c} = b$.

5. Also, if any two products be equal to each other, either of the terms of one of them, will be to either of the terms of the other, as the remaining term of the last is to the remaining term of the first.

Thus, if $ad = bc$, or $2 \times 15 = 6 \times 5$, then will any of the following forms of these quantities be proportional:

Directly,

$$a : b :: c : d, \text{ or } 2 : 6 :: 5 : 15.$$

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tities be required, they may be expressed in algebraical terms, as below:

$\sqrt[3]{a^2b}$ and $\sqrt[3]{ab^2} =$ two geometrical means between a and b.

$\sqrt[4]{a^3b}$, $\sqrt[4]{a^2b^2}$, and $\sqrt[4]{ab^3} =$ three geometrical means between a and b.

$(a^nb)^{\frac{1}{n+1}}$, $(a^{n-1}b^2)^{\frac{1}{n+1}}$, $(a^{n-2}b^3)^{\frac{1}{n+1}}$, &c. to $(ab^n)^{\frac{1}{n+1}} =$ any number (n) of geometrical means between a and b.

Invertedly,

$b : a :: d : c$, or $6 : 2 :: 15 : 5$.

Alternately,

$a : c :: b : d$, or $2 : 5 :: 6 : 15$.

Conjunctly,

$a : a+b :: c : c+d$, or $2 : 8 :: 5 : 20$.

Disjunctly,

$a : b-a :: c : d-c$, or $2 : 4 :: 5 : 10$.

Mixedly,

$b+a : b-a :: d+c : d-c$, or $8 : 4 :: 20 : 10$.

In all of which cases, the product of the two extremes is equal to that of the two means (*l*).

6. It also follows, as a consequence of the general equality between the products of the means and extremes, that any like parts, multiples, roots, or powers, of four proportional quantities, are proportional.

Thus, if $a : b :: c : d$,

$$\text{Then,}\left|\begin{array}{l} a : b :: nc : nd; \\ na : nb :: c : d, \\ na : nb :: mc : md, \\ a^m : b^m :: c^m : d^m. \end{array}\right.$$

Where *m* and *n* may be any whole or fractional numbers whatever.

(*l*) The theory of proportion was, for a long time, encumbered with a variety of Latin terms, denoting the different changes and transformations that ratios might be made to undergo, without altering the relation of their terms; but as these are, in general, of little use, besides being very burdensome to the memory, the tendency, of late, has been to discard them, and to substitute the corresponding equations in their stead; which is certainly a more commodious method. I have, therefore, inserted such of these appellations only as are still employed in geometry, where they cannot be easily dispensed with.

Hence, likewise, in any proportion, if instead of either of the antecedents and its consequent, any quantities, that have the same ratio, be substituted, the proportionality will still subsist.

Thus, if $a : b :: c : d$, and $c : d :: e : f$; then

$$a : b :: e : f.$$

7. In like manner, if the corresponding terms of two, or more, ranks of proportionals be multiplied together, the resulting terms will still be proportional.

Thus, if $a : b :: c : d$, and $e : f :: g : h$; then

$$ae : bf :: cg : dh.$$

8. Also, if four quantities be proportional, their reciprocals will be proportional; and if the reciprocals of four quantities be proportional, the quantities themselves will be proportional.

Thus, if $2 : 6 :: 3 : 9$, or $a : b :: c : d$; then will

$$\frac{1}{2} : \frac{1}{6} :: \frac{1}{3} : \frac{1}{9}, \text{ and } \frac{1}{a} : \frac{1}{b} :: \frac{1}{c} : \frac{1}{d}$$

9. In any continued geometrical series, the product of the two extremes is equal to the product of any two means that are equally distant from them; or to the square of the mean, when the number of terms is odd.

Thus, if the series be 2, 4, 8, 16, 32; then

$$2 \times 32 = 4 \times 16 = 8^2.$$

10. If any number of quantities be in geometrical progression, their differences will also be in geometrical progression; and the quantities will be proportional to their differences.

Thus, in the series a, ar, ar^2, ar^3, ar^4, &c. their differences $ar - a$, $ar^2 - ar$, $ar^3 - ar^2$, $ar^4 - ar^3$, &c. form a progression, of which the ratio is r.

Also,

$a : ar :: ar - a : ar^2 - ar :: ar^2 - ar : ar^3 - ar^2$, &c. the ratio of all the corresponding terms being r, as before.

11. In any geometrical series, the last term is equal to the product arising from multiplying the first term by such a power of the ratio as is denoted by the number of terms less one.

Thus, in the series 2, 6, 18, 54, 162, we shall have $2 \times 3^4 = 2 \times 81 = 162$.

And in the series a, ar, ar^2, ar^3, ar^4, &c. continued to n terms, we shall have $a \times r^{n-1} = ar^{n-1}$.

12. The sum of any series of quantities in geometrical progression, either increasing or decreasing, is found by multiplying the last term by the ratio, and then dividing the difference of this product and the first term by the difference between the ratio and unity.

Thus, in the series 2, 4, 8, 16, 32, 64, 128, 256, 512, we shall have $\frac{512 \times 2 - 2}{2 - 1} = 1024 - 2 = 1022$, the sum.

Or the same rule, without considering the last term, may be expressed thus:

Find such a power of the ratio as is denoted by the number of terms of the series; then divide the difference between this power and 1 by the difference between the ratio and 1, and the result, multiplied by the first term, will be the sum of the series.

Thus, in the series $a + ar + ar^2 + ar^3 + ar^4$, &c. to ar^{n-1}, we have $\frac{ar^{n-1} \times r - a}{r - 1} = \frac{ar^n - a}{r - 1} = a\left(\frac{r^n - 1}{r - 1}\right)$ = the sum.

Where it is to be observed, that if r, in the last of these progressions, be a proper fraction, and consequently the series a decreasing one, the sum of $a+ar+ar^2+ar^3+ar^4$, &c. continued *ad infinitum*, will be $\frac{a}{1-r}$

13. Three quantities are said to be in harmonical proportion, when the first has the same ratio to the third, as the difference between the first and second has to the difference between the second and third.

Thus, a, b, c, are harmonically proportional, when $a : c :: a-b : b-c$.

14. And four quantities are in harmonical proportion, when the first has the same ratio to the fourth, as the difference between the first and second has to the difference between the third and fourth.

Thus, the quantities a, b, c, d, are in harmonical proportion, when $a : d :: a-b : c-d$.

Also, if there be taken any series of arithmeticals a, $a+d$, $a+2d$, $a+3d$, &c. their reciprocals $\frac{1}{a}$, $\frac{1}{a+d}$, $\frac{1}{a+2d}$, $\frac{1}{a+3d}$, &c. will be harmonicals; and the contrary (m).

(m) In addition to these properties, it may be remarked, that an harmonical mean between any two quantities, is equal to twice their product divided by their sum.

Thus, $\frac{2ab}{a+b}$ = an harmonical mean between a and b.

It may here, also, be farther observed, that the ratio of two squares is frequently called *duplicate ratio*; of two square roots, *subduplicate ratio*; of two cubes, *triplicate ratio*; and of two cube roots, *subtriplicate ratio*, &c.

One ratio is likewise said to be greater or less than another,

EXAMPLES.

1. The first term of a geometrical series is 1, the ratio 2, and the number of terms 10; what is the sum of the series?

First, $1 \times 2^9 = 1 \times 512 = 512$, the last term.

And $\frac{512 \times 2 - 1}{2 - 1} = \frac{1024 - 1}{1} = 1023$, the sum required.

2. The first term of a geometrical series is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 5; required the sum of the series.

First, $\frac{1}{2} \times (\frac{1}{3})^4 = \frac{1}{2} \times \frac{1}{81} = \frac{1}{162}$, the last term.

And $\frac{\frac{1}{2} - \frac{1}{162} \times \frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{2} - \frac{1}{486}}{\frac{2}{3}} = \frac{121}{243} \times \frac{3}{2} = \frac{121}{162}$, the sum.

3. Required the sum of 1, 3, 9, 27, 81, &c. continued to 12 terms. Ans. 265720.

4. Required the sum of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, &c. continued to 10 terms. Ans. $1\frac{3280}{6561}$.

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according as its antecedent is a greater or less multiple, or part, of its consequent.

Thus, the ratio of 7 to 4 is greater than that of 8 to 5, because $\frac{7}{4}$, or $\frac{35}{20}$, is greater than $\frac{8}{5}$, or $\frac{32}{20}$; and the ratio of 2 to 3 is less than that of 3 to 4, because $\frac{2}{3}$, or $\frac{8}{12}$, is less than $\frac{3}{4}$, or $\frac{9}{12}$.

To this we may also add, that if any three of the quantities a, r, n, s, be given, the fourth may be found from the equation

$$s = \frac{ar^n - a}{r - 1}, \text{ or } \frac{rl - a}{r - 1}$$

Where the last term of the progression l is $= ar^{n-1}$, as above.

5. Required the sum of 1, 2, 4, 8, 16, 32, &c. continued to 100 terms. Ans. 1048575.

6. A person being asked to dispose of a fine horse, said he would sell him on condition of having a farthing for the first nail in his shoes, a halfpenny for the second, a penny for the third, twopence for the fourth, and so on, doubling the price of every nail, to 32, the number of nails in his four shoes; what would the horse be sold for at that rate? Ans. 4473924*l.* 5*s.* 3$\frac{3}{4}$*d.*

OF EQUATIONS.

(N) THE DOCTRINE OF EQUATIONS is that branch of algebra, which treats of the methods of determining the values of unknown quantities by means of their relations to others which are known.

This is done by making certain algebraic expressions equal to each other; which formula, in that case, is called an equation; and then working by the rules of the art, till the quantity sought is found equal to some given quantity, and consequently becomes known.

The terms of an equation are the quantities of which it is composed; and the parts, that stand on the right and left of the sign =, are called the two members, or sides, of the equation.

Thus, if $x=a+b$, the terms are x, a, and b; and the meaning of the expression is, that some quantity x, standing on the left hand side of the equation, is equal to the sum of the quantities a and b on the right hand side.

A simple equation is that, which contains only one power of the unknown quantity: as, $x+a=3b$, or $ax^2=bc$, or $2x^3+3a^2=5b^2$; where x denotes the unknown quantity, and the other letters the known quantities.

A compound equation is that, which contains two or more different powers of the unknown quantity; as, $x^2+ax=b$, or $x^3-4x^2+3x=25$.

Equations are also divided into different orders, or receive particular names, according to the highest power of the unknown quantity contained in any one of their terms: as, quadratic equations, cubic equations, biquadratic equations, &c.

Thus, a quadratic equation is that, in which the unknown quantity is of two dimensions, or which rises to the second power: as, $x^2=20$; $x^2+ax=b$; or $3x^2+10x=100$.

A cubic equation is that, in which the unknown quantity is of three dimensions, or which rises to the third power: as, $x^3=27$; $2x^3-3x=35$; or $x^3-ax^2+bx=c$.

A biquadratic equation is that, in which the unknown quantity is of four dimensions, or which rises to the fourth power: as, $x^4=25$; $5x^4-4x=6$; or $x^4-ax^3+bx^2-cx=d$.

And so on, for equations of the 5th, 6th, 7th, &c. order, which are all denominated according to the highest power of the unknown quantity contained in any one of their terms.

The root of an equation is such a number, or quantity, as, being substituted for the unknown quantity, will make both sides of the equation vanish, or become equal to each other.

A simple equation can have only one root; but every compound equation has as many roots as it contains dimensions, or as is denoted by the index of the highest power contained in it.

Thus, in the quadratic equation $x^2+2x=15$, the root, or value of x, is either $+3$ or -5; and, in the cubic equation $x^3-9x^2+26x=24$, the roots are 2, 3, and 4, as will be found by substituting each of these numbers for x.

OF THE RESOLUTION OF SIMPLE EQUATIONS, CONTAINING ONLY ONE UNKNOWN QUANTITY.

(o) The resolution of simple, as well as of other equations, is the disengaging the unknown quantity, in all such expressions, from the other quantities with which it is connected, and making it stand alone, on one side of the equation, so as to be equal to such as are known on the other side; for the performing of which, several axioms and processes are required, the most useful and necessary of which are the following (*n*):

(*n*) The operations required, for the purpose here mentioned, are chiefly such as are derived from the following simple and evident principles:

1. If the same quantity be added to, or subtracted from, each of two equal quantities, the results will still be equal; which is the same, in effect, as taking any quantity from one side of an equation, and placing it on the other, with a contrary sign.
2. If all the terms of any two equal quantities, be multiplied, or divided, by the same quantity, the products, or quotients, thence arising, will be equal.

CASE I.

Any quantity may be transposed from one side of an equation to the other, by changing its sign; and the two members, or sides, will be still equal.

Thus, if $x+3=7$; then will $x=7-3$, or $x=4$.

And, if $x-4+6=8$; then will $x=8+4-6=6$.

Also, if $x-a+b=c-d$; then will $x=a-b+c-d$.

And, if $4x-8=3x+20$; then $4x-3x=20+8$, or $x=28$.

From this rule it also follows, that if a quantity be found on each side of an equation, with the same sign, it may be left out of them both; and that the signs of all the terms, of any equation, may be changed from + to −, or from − to +, without altering its value.

Thus, if $x+5=7+5$; then, by cancelling, $x=7$.

And, if $a-x=b-c$; then $x-a=c-b$, or $x=a+c-b$.

CASE II.

If the unknown quantity, in any equation, be multiplied by any quantity, the multiplier may be taken away by dividing all the rest of the terms by it; and if it be divided by any quantity, the divisor may be taken away by multiplying all the other terms by it.

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3. If two quantities, either simple or compound, be equal to each other, any like powers, or roots, of them will also be equal.

All of which axioms will be found sufficiently illustrated, by the processes arising out of the several examples annexed to the six different cases given in the text.

Thus, if $ax=3ab-c$; then will $x=3b-\frac{c}{a}$.

And, if $2x+4=16$; then will $x+2=8$, or $x=8-2=6$.

Also, if $\frac{x}{2}=5+3$; then will $x=10+6=16$.

And, if $\frac{2x}{3}-2=4$; then $2x-6=12$, or $x-3=6$, or $x=9$.

CASE III.

If either of the terms of an equation be a fraction, its denominator may be taken away by multiplying all the rest of the terms by it; or, if several of them be fractions, they may be reduced to integers by multiplying both sides of the equation by the product of their denominators, or by the least common multiple of them.

Thus, if $2x+\frac{3x}{4}=22$; then will $8x+3x=88$, or $11x=88$, or $x=\frac{88}{11}=8$.

And, if $\frac{x}{4}+\frac{x}{6}=10$; then, multiplying by 12, (which is a multiple of 4 and 6,) $3x+2x=120$, or $5x=120$, or $x=\frac{120}{5}=24$.

It also appears, from this rule, that if the same number, or letter, be found in each of the terms of an equation, either as a multiplier or divisor, it may be expunged from them all, without altering the result.

Thus, if $ax=ab+ac$; then, by cancelling, $x=b+c$.

And, if $\frac{x}{a}+\frac{b}{a}=\frac{c}{a}$; then $x+b=c$, or $x=c-b$.

CASE IV.

If the unknown quantity, in any equation, be in the form of a surd, let it be made to stand alone, on one side of the equation, and put all the remaining terms on the other, (by Case 1); then involve each of the sides to such a power as corresponds with the index of the surd, and the equation will be rendered free from any irrational expression.

Thus, if $\sqrt{x}-2=3$; then will $\sqrt{x}=3+2=5$, or $x=5^2=25$.

And, if $\sqrt{3x+4}=5$; then will $3x+4=25$, or $3x=25-4=21$, or $x=\frac{21}{3}=7$.

Also, if $\sqrt[3]{2x+3}+4=8$; then will $\sqrt[3]{2x+3}=8-4=4$, or $2x+3=4^3=64$, or $2x=64-3=61$, or $x=\frac{61}{2}=30\frac{1}{2}$

CASE V.

If that side of the equation, which contains the unknown quantity, be a complete power, the equation may be reduced to a lower dimension by extracting the root of the said power, on both sides of the equation.

Thus, if $x^2=81$; then $x=\sqrt{81}=9$; and if $x^3=27$, then $x=\sqrt[3]{27}=3$.

Also, if $3x^2-9=24$; then $3x^2=24+9=33$, or $x^2=\frac{33}{3}=11$, or $x=\sqrt{11}$.

And, if $x^2+6x+9=27$; then, since the left hand side of the equation is a complete square, we shall have, by extracting the roots, $x+3=\sqrt{27}=3\sqrt{3}$, or $x=3\sqrt{3}-3$.

CASE VI.

Any analogy, or proportion, may be converted into an equation, by making the product of the two extreme terms equal to that of the two means.

Thus, if $3x : 16 :: 5 : 6$; then $3x \times 6 = 16 \times 5$, or $18x = 80$, or $x = \frac{80}{30} = \frac{8}{3} = 2\frac{2}{3}$.

And, if $\frac{2x}{3} : a :: b : c$; then will $\frac{2cx}{3} = ab$, or $2cx = 3ab$, or $x = \frac{3ab}{2c}$.

Also, if $12 - x : \frac{x}{2} :: 4 : 1$; then $12 - x = \frac{4x}{2} = 2x$, or $2x + x = 12$, or $x = \frac{12}{3} = 4$.

MISCELLANEOUS EXAMPLES.

1. Given $5x - 15 = 2x + 6$, to find the value of x.

Here $5x - 2x = 6 + 15$, or $3x = 6 + 15 = 21$; and therefore $x = \frac{21}{3} = 7$.

2. Given $40 - 6x - 16 = 120 - 14x$, to find the value of x.

Here $14x - 6x = 120 - 40 + 16$; or $8x = 136 - 40 = 96$; and therefore $x = \frac{96}{8} = 12$.

3. Given $3x^2 - 10x = 8x + x^2$, to find the value of x.

Here $3x - 10 = 8 + x$, by dividing by x; or $3x - x = 8 + 10 = 18$; and therefore $2x = 18$, or $x = \frac{18}{2} = 9$.

4. Given $6ax^3 - 12abx^2 = 3ax^3 + 6ax^2$, to find the value of x.

Here $2x - 4b = x + 2$, by dividing by $3ax^2$; or $2x - x = 2 + 4b$; and therefore $x = 4b + 2$.

5. Given $x^2+2x+1=16$, to find the value of x.

Here $x+1=4$, by extracting the roots; and therefore $x=4-1=3$.

6. Given $5ax-3b=2dx+c$, to find the value of x.

Here $5ax-2dx=c+3b$; or $(5a-2d)x=c+3b$; and therefore $x=\frac{c+3b}{5a-2d}$.

7. Given $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=10$, to find the value of x.

Here $x-\frac{2x}{3}+\frac{2x}{4}=20$; or $3x-2x+\frac{6x}{4}=60$; or $12x-8x+6x=240$; whence $10x=240$, or $x=24$.

8. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$, to find the value of x.

Here $x-3+\frac{2x}{3}=40-x+19$; or $3x-9+2x=120-3x+57$; whence $3x+2x+3x=120+57+9$; that is $8x=186$, or $x=23\frac{1}{4}$.

9. Given $\sqrt{\frac{2x}{3}}+5=7$, to find the value of x.

Here $\sqrt{\frac{2x}{3}}=7-5=2$; whence $\frac{2x}{3}=2^2=4$, and $2x=12$, or $x=6$.

10. Given $x+\sqrt{(a^2+x^2)}=\frac{2a^2}{\sqrt{(a^2+x^2)}}$, to find the value of x.

Here $x\sqrt{(a^2+x^2)}+a^2+x^2=2a^2$; whence $x\sqrt{(a^2+x^2)}=a^2-x^2$, and $x^2(a^2+x^2)=a^4-2a^2x^2+x^4$, or $a^2x^2+x^4=a^4-2a^2x^2+x^4$, or $a^2x^2=a^4-2a^2x^2$, or $3a^2x=a^4$, or $x^2=\frac{a^4}{3a^2}=\frac{a^2}{3}$; and therefore $x=\sqrt{\frac{a^2}{3}}=a\sqrt{\frac{1}{3}}=a\sqrt{\frac{3}{9}}=\frac{a}{3}\sqrt{3}$.

EXAMPLES FOR PRACTICE.

1. Given $3x-2+24=31$, to find the value of x.
Ans. $x=3$

2. Given $6-2y+10=20-3y-2$, to find the value of y.
Ans. $y=2$

3. Given $x+18=3x-5$, to find the value of x.
Ans. $x=11\frac{1}{2}$

4. Given $x+\frac{x}{2}+\frac{x}{3}=11$, to find the value of x.
Ans. $x=6$

5. Given $2x-\frac{x}{2}+1=5x-2$, to find the value of x.
Ans. $x=\frac{6}{7}$

6. Given $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{7}{10}$, to find the value of x.
Ans. $x=1\frac{1}{5}$

7. Given $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}$, to find the value of x.
Ans. $x=3\frac{6}{13}$

8. Given $\sqrt{12+x}=6+\sqrt{x}$, to find the value of x.
Ans. $x=4$

9. Given $x+a=\frac{x^2}{a+x}$, to find the value of x.
Ans. $x=-\frac{a}{2}$

10. Given $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{(a+x)}}$, to find the value of x.
Ans. $x=\frac{a}{3}$

11. Given $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$, to find the value of x.
Ans. $x=\frac{3b}{3a-2b}$

12. Given $\sqrt{a^2+x^2} = \sqrt[4]{b^4+x^4}$, to find the value of x.

Ans. $x=\sqrt{\frac{b^4-a^4}{2a^2}}$

13. Given $\sqrt{a+x}+\sqrt{a-x}=\sqrt{ax}$, to find the value of x. Ans. $x=\frac{4a^2}{a^2+4}$

14. Given $\frac{a}{1+x}+\frac{a}{1-x}=b$, to find the value of x.

Ans. $x=\sqrt{\frac{b-2a}{b}}$

15. Given $a+x=\sqrt{a^2+x\sqrt{(b^2+x^2)}}$, to find the value of x. Ans. $x=\frac{b^2}{4a}-a$

16. Given $\frac{1}{2}\sqrt{x^2+3a^2}-\frac{1}{2}\sqrt{x^2-3a^2}=x\sqrt{a}$, to find the value of x. Ans. $x=\sqrt[4]{\frac{9a^3}{4-4a}}$

17. Given $\sqrt{a+x}+\sqrt{a-x}=b$, to find the value of x. Ans. $x=\frac{b}{2}\sqrt{4a-b^2}$

18. Given $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b$, to find the value of x. Ans. $x=\sqrt{a^2-\left(\frac{b^3-2a}{3b}\right)^3}$

19. Given $\sqrt{a}+\sqrt{x}=\sqrt{ax}$, to find the value of x.

Ans. $x=\frac{a}{(\sqrt{a}-1)^2}$

20. Given $\sqrt{a^2-ax}=a-\sqrt{a-ax}$, to find the value of x. Ans. $x=\frac{2a\sqrt{a}+a}{(\sqrt{a}+1)^2}$

21. Given $\sqrt{x+a}=c-\sqrt{x+b}$, to find the value of x.

Ans. $x=\left(\frac{c^2+b-a}{2c}\right)^2-b$

22. Given $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt[4]{\frac{4bc}{a^2-x^2}}$, to find the value of x. Ans. $x=a\left(\frac{b+c}{b-c}\right)$

Of the resolution of simple equations, containing two unknown quantities.

When there are two unknown quantities, and two independent simple equations, involving them, they may be reduced to one, by any of the three following rules;

RULE I.

Observe which of the unknown quantities is the least involved, and find its value in each of the equations, by the methods already explained; then let the two values, thus found, be put equal to each other, and there will arise a new equation with only one unknown quantity in it, the value of which may be found as before(*o*).

EXAMPLES.

1. Given $\left\{\begin{matrix} 2x+3y=23 \\ 5x-2y=10 \end{matrix}\right\}$ to find the values of x and y.

From the first equation, $x=\frac{23-3y}{2}$,

And from the second, $x=\frac{10+2y}{5}$,

Whence we have $\frac{23-3y}{2}=\frac{10+2y}{5}$,

Or $115-15y=20+4y$, or $19y=115-20=95$,

That is, $y=\frac{95}{19}=5$, and $x=\frac{23-15}{2}=4$.

(*o*) This rule depends upon the well known axiom, that things which are equal to the same thing, are equal to each other; and the two following methods are founded on principles which are equally simple and obvious.

2. Given $\left\{\begin{matrix} x+y=a \\ x-y=b \end{matrix}\right\}$ to find the values of x and y.

From the first equation, $x=a-y$,
And from the second, $x=b+y$,
Whence $a-y=b+y$, or $2y=a-b$,
Consequently $y=\frac{a-b}{2}$, and $x=a-y$,
Or, by substitu., $x=a-\frac{a-b}{2}=\frac{a+b}{2}$.

3. Given $\left\{\begin{matrix} \frac{1}{2}x+\frac{1}{3}y=7 \\ \frac{1}{3}x+\frac{1}{2}y=8 \end{matrix}\right\}$ to find the values of x and y.

From the first equation, $x=14-\frac{2y}{3}$,
And from the second, $x=24-\frac{3y}{2}$,
Consequently $14-\frac{2y}{3}=24-\frac{3y}{2}$,
And therefore $42-2y=72-\frac{9y}{2}$,
Or, by multi., $84-4y=144-9y$;
Whence also $5y=144-84=60$,
Or $y=\frac{60}{5}=12$, and $x=14-\frac{24}{3}=6$.

RULE II.

Find the value of either of the unknown quantities in that equation in which it is the least involved; then substitute this value for its equal in the other equation, and there will arise a new equation with only one unknown quantity in it; the value of which may be found as before.

EXAMPLES.

1. Given $\left\{\begin{matrix} x+2y=17 \\ 3x-y=2 \end{matrix}\right\}$ to find the values of x and y.

From the first equation, $x=17-2y$; and this value, being substituted for x, in the second,

$$\text{Gives}\quad 3(17-2y)-y=2,$$

$$\text{Or}\quad 51-6y-y=2,\quad\text{or}\quad 7y=51-2=49,$$

$$\text{Whence } y=\frac{49}{7}=7,\quad\text{and}\quad x=17-2y=3.$$

2. Given $\left\{\begin{matrix} x+y=13 \\ x-y=3 \end{matrix}\right\}$ to find the values of x and y.

From the first equation, $x=13-y$; and this value being substituted for x, in the second,

$$\text{Gives}\quad 13-y-y=3,\text{ or } 2y=13-3=10,$$

$$\text{Hence } y=\frac{10}{2}=5,\quad\text{and}\quad x=13-y=8.$$

3. Given $\left\{\begin{matrix} x:y::a:b \\ x^2+y^2=c \end{matrix}\right\}$ to find the values of x and y.

Here the first analogy, turned into an equation,

$$\text{Gives } bx=ay,\text{ or } x=\frac{ay}{b},$$

And this value, substituted for x in the second,

$$\text{Gives}\quad \left(\frac{ay}{b}\right)^2+y^2=c,\text{ or } \frac{a^2y^2}{b^2}+y^2=c,$$

$$\text{Therefore } a^2y^2+b^2y^2=b^2c,\text{ or } y^2=\frac{b^2c}{a^2+b^2},$$

$$\text{Hence } y=b\sqrt{\frac{c}{a^2+b^2}},\text{ and } x=a\sqrt{\frac{c}{a^2+b^2}}.$$

RULE III.

Let the given equations be multiplied, or divided, by such numbers, or quantities, as will make the term that contains one of the unknown quantities the same in them both; then, by adding, or subtracting, the two equations thus obtained, as the case may require, there will arise a new equation, with only one unknown quantity in it, which may be resolved as before (*p*).

EXAMPLES.

1. Given $\left\{\begin{array}{r} 3x+5y=40 \\ x+2y=14 \end{array}\right\}$ to find the values of x and y.

First, multiply the second equation by 3, and it will give $3x+6y=42$.

Then, subtract the first equation from this, and it will give $6y-5y=42-40$, or $y=2$.

Whence, also, $x=14-2y=14-4=10$.

2. Given $\left\{\begin{array}{r} 5x-3y=9 \\ 2x+5y=16 \end{array}\right\}$ to find the values of x and y.

Multiply the first equation by 2, and the second by 5; then $10x-6y=18$, and $10x+25y=80$.

And if the former of these be subtracted from the latter it will give $31y=62$, or $y=\frac{62}{31}=2$.

And, by the first equation, $x=\frac{9+3y}{5}=\frac{15}{5}=3$.

(*p*) The values of the unknown quantities in the two literal equations

$$ax+by=c,\ a'x+b'y=c',$$

may be readily found, in general terms, by multiplying the first by a', and the second by a, and then working according to the rule; the results, so determined, being

$$y=\frac{ac'-ca'}{ab'-ba'},\ x=\frac{cb'-bc'}{ab'-ba'}.$$

Which solution may be applied to any particular case of this kind; by substituting the numeral values of a, b, a', b', in the place of the letters, and observing, when either of them is negative, to change the signs accordingly.

EXAMPLES FOR PRACTICE.

1. Given $4x+y=34$, and $4y+x=16$, to find the values of x and y. Ans. $x=8$, $y=2$

2. Given $2x+3y=16$, and $3x-2y=11$, to find the values of x and y. Ans. $x=5$, $y=2$

3. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$ to find the values of x and y. Ans. $x=\frac{1}{2}$, $y=\frac{1}{3}$

4. Given $\frac{x}{7}+7y=99$, and $\frac{y}{7}+7x=51$, to find the values of x and y. Ans. $x=7$, $y=14$

5. Given $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$, to find the values of x and y.

Ans. $x=60$, $y=40$

6. Given $x+y=s$, and $x^2-y^2=d$, to find the values of x and y. Ans. $x=\frac{s^2+d}{2s}$, $y=\frac{s^2-d}{2s}$

7. Given $x+y : a :: x-y : b$, and $x^2-y^2=c$, to find the values of x and y.

Ans. $x=\frac{a+b}{2}\sqrt{\frac{c}{ab}}$, $y=\frac{a-b}{2}\sqrt{\frac{c}{ab}}$

8. Given $ax+by=c$, and $dx+ey=f$, to find the values of x and y.

Ans. $x=\frac{ce-bf}{ae-bd}$, $y=\frac{af-dc}{ae-bd}$

9. Given $x+y=a$, and $x^2+y^2=b$, to find the values of x and y. Ans. $x=\frac{a^2+b}{2a}$, $y=\frac{a^2-b}{2a}$

10. Given $x^2+xy=a$, and $y^2+xy=b$, to find the values of x and y. Ans. $x=\frac{a}{\sqrt{a+b}}$, $y=\frac{b}{\sqrt{a+b}}$

Of the resolution of simple equations, containing three unknown quantities.

When there are three unknown quantities, and three independent simple equations containing them, they may be reduced to one, by the following method (*q*).

RULE.

Find the values of one of the unknown quantities, in each of the three given equations, as if all the rest were known; then put the first of these values equal to the second, and either the first or second equal to the third, and there will arise two new equations with only two unknown quantities in them, the values of which may be found as in the former case; and thence the value of the third.

Or, multiply each of the three equations by such numbers as will make one of their terms the same in them all; then, having subtracted any two of

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(*q*) The necessity for observing that the given equations, in this and other similar cases, are so proposed as to be independent of each other, will be obvious from the following example:

$$x-2y+z=5;\ 2x+y-z=7;\ x+3y-z=2;$$

where, if it were required to determine the values of x, y, and z, it will be found, by eliminating x from each of them, and then equating the two results, that

$$5y-3z=-3, \text{ and } 5y-3z=-3;$$

which equations, being identical, or both the same, furnish no determinate answer. And, in effect, if the three equations be properly examined, it will be found, that the third is merely the difference of the first and second, and consequently involves no condition but what is contained in the other two.

these resulting equations from the third, or added them together, as the case may require, there will remain only two equations, which may be resolved by the former rules.

And in nearly the same way may any number of unknown quantities be exterminated; but, in cases of this kind, there are frequently shorter and more commodious methods of operation, which can only be learnt from practice (*r*).

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=29 \\ x+2y+3z=62 \\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=10 \end{array}\right\}$ to find x, y, and z.

From the first equa., $x=29-7-z$,

From the second, $x=62-2y-3z$,

And from the third, $x=20-\frac{2}{3}y-\frac{1}{2}z$,

Whence $29-y-z=62-2y-3z$,

And $29-y-z=20-\frac{2}{3}y-\frac{1}{2}z$.

(*r*) The values of the unknown quantities in the three literal equations

$$ax+by+cz=d;\ a'x+b'y+c'z=d';\ a''x+b''y+c''z=d'';$$

may be exhibited, in general terms, like those before mentioned, as follows:

$$x=\frac{db'c''-dc'b''+cd'b''-bd'c''+bc'd''-cb'd''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}$$

$$y=\frac{ad'c''-ac'd''+ca'd''-da'c''+dc'a''-cd'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}$$

$$z=\frac{ab'd''-ad'b''+da'b''-ba'd''+bd'a''-db'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}$$

Which formulæ, by substitution, may be employed for the resolution of any numeral case of this kind, as in the instance of two equations before given.

Also, from the first of these, $y=33-2z$,

And from the second, $y=27-\frac{3}{2}z$,

Therefore $33-2z=27-\frac{3}{2}z$, or $z=12$,

Whence, also, $y=33-2z=9$,

And $x=29-y-z=8$.

2. Given $\left\{\begin{array}{l} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-z\ \ =63 \end{array}\right\}$ to find x, y, and z.

Here, multiplying the first equation by 6, the second by 3, and the third by 2, we shall have

$$\begin{aligned} 12x+24y-18z&=132, \\ 12x-\ \ 6y+15z&=54, \\ 12x+14y-\ \ 2z&=126. \end{aligned}$$

And, subtracting the second of these equations successively from the first and third, there will arise

$$\begin{aligned} 30y-33z&=78, \\ 20y-17z&=72. \end{aligned}$$

Or, by dividing the first of these two equations by 3, and then multiplying the result by 2,

$$\begin{aligned} 20y-22z&=52, \\ 20y-17z&=72. \end{aligned}$$

Whence, by subtracting the former of these from the latter, $5z=20$, or $z=4$.

And, consequently, by substitution,

$$y=7 \text{ and } x=3.$$

3. Given $x+y+z=53$, $x+2y+3z=105$, and $x+3y+4z=134$, to find the values of x, y, and z.

Ans. $x=24$, $y=6$, and $z=23$

4. Given $x+\frac{1}{2}y+\frac{1}{3}z=32$, $\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15$, and $\frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12$, to find the values of x, y, and z.

Ans. $x=12$, $y=20$, $z=30$

5. Given $7x+5y+2z=79$, $8x+7y+9z=122$, and $x+4y+5z=55$, to find the values of x, y, and z.

Ans. $x=4$, $y=9$, $z=3$

6. Given $x+y=a$, $x+z=b$, and $y+z=c$, to find the values of x, y, and z.

MISCELLANEOUS QUESTIONS, PRODUCING SIMPLE EQUATIONS.

The usual method of resolving algebraical questions, is first to denote the quantities that are to be found by x, y, or some other letter of the alphabet; then, having properly examined the state of the question, perform with these letters, and the known quantities, by means of the common signs, the same operations and reasonings, that it would be necessary to make if the quantities were known, and it was required to verify them, and the conclusion will give the result sought.

Or, it is sometimes best to denote only one of the unknown quantities by x or y, and then to determine the expression for the other from the nature of the question; after which the same method of reasoning may be followed, as above. And, in some cases, the substituting an unknown letter for the sum, or difference, &c. of quantities, instead of the quantities themselves, will facilitate the solution.

1. What number is that whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 16?

Let $x=$ the number required,

Then its $\frac{1}{3}$ part will be $\frac{1}{3}x$, and its $\frac{1}{4}$ part $\frac{1}{4}x$, And therefore $\frac{1}{3}x - \frac{1}{4}x = 16$, by the question,

That is $x - \frac{3}{4}x = 48$, or $4x - 3x = 192$,

Hence $x = 192$, the number required.

2. It is required to find two numbers such, that their sum shall be 40, and their difference 16.

Let x denote the least of the two numbers required,

Then will $x + 16 =$ to the greater number,

And $x + x + 16 = 40$, by the question,

That is $2x = 40 - 16$, or $x = \frac{24}{2} = 12 =$ least num.,

And $x + 16 = 12 + 16 = 28 =$ the greater number required.

3. Divide 1000*l.* between A, B, and C, so that A shall have 72*l.* more than B, and C 100*l.* more than A.

Let $x =$ B's share of the given sum,

Then will $x + 72 =$ A's share,

And $x + 172 =$ C's share.

Hence their sum is $x + x + 72 + x + 172$,

Or $3x + 244 = 1000$, by the question,

That is $3x = 1000 - 244 = 756$,

Or $x = \frac{756}{3} = 252l. =$ B's share,

Hence $x + 72 = 324l. =$ A's share,

And $x + 172 = 424l. =$ C's share.

Sum of all $= 1000l.$ the proof.

4. It is required to divide 1000*l.* between two persons, so that their shares of it shall be in the proportion of 7 to 9.

Let $x =$ the first person's share,

Then will $1000 - x =$ second person's share, And $x : 1000-x :: 7 : 9$, by the question,

That is $9x = \overline{1000-x} \times 7 = 7000 - 7x$,

Or $9x + 7x = 7000$, or $x = \frac{7000}{16} = 437l.\ 10s.$ = first share, and $1000 - x = 1000 - 437l.\ 10s. = 562l.\ 10s.$ = second share.

5. The paving of a square court with stones, at 2*s*. a yard, will cost as much as the enclosing it with pallisades, at 5*s*. a yard; required the side of the square?

Let $x =$ length of the side of the square sought,

Then $4x =$ number of yards of enclosure,

And $x^2 =$ number of yards of pavement,

Hence $4x \times 5 = 20x =$ price of enclosing,

And $x^2 \times 2 = 2x^2 =$ price of paving,

Therefore $2x^2 = 20x$, by the question,

Or $2x = 20$, and $x = 10$, the length of the side required.

6. Out of a cask of wine, which had leaked away $\frac{1}{3}$ part, 21 gallons were afterwards drawn, and the cask being then guaged, appeared to be half full; how much did it hold?

Let $x =$ the number of gallons the cask is supposed to have held,

Then it would have leaked away $\frac{1}{3}x$ gallons,

Whence there had been taken out of it, altogether, $21 + \frac{1}{3}x$ gallons,

And therefore $21 + \frac{1}{3}x = \frac{1}{2}x$, by the question,

That is $63 + x = \frac{3}{2}x$, or $126 + 2x = 3x$,

Consequently $3x - 2x = 126$, or $x = 126$, the number of gallons required.

7. What fraction is that, to the numerator of which if 1 be added, its value will be $\frac{1}{3}$, but if 1 be added to the denominator, its value will be $\frac{1}{4}$.

Let the fraction be represented by $\frac{x}{y}$,

Then $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$, by the question,

Hence $3x+3=y$, and $4x=y+1$, or $x=\frac{y+1}{4}$,

Therefore $3\left(\frac{y+1}{4}\right)+3=y$, or $3y+3+12=4y$,

That is $y=15$, and $x=\frac{y+1}{4}=\frac{15+1}{4}=\frac{16}{4}=4$,

Whence the fraction required is $\frac{4}{15}$.

8. A market woman bought in a certain number of eggs at 2 a penny, and as many others at 3 a penny, and having sold them out again, altogether, at the rate of 5 for 2*d*., found she had lost 4*d*.; how many eggs had she?

Let $x=$ the number of eggs of each sort,

Then will $\frac{1}{2}x=$ the price of the first sort,

And $\frac{1}{3}x=$ the price of the second sort,

But $5:2::2x$ (the whole number of eggs) $:\frac{4x}{5}$,

Whence $\frac{4x}{5}=$ the price of both sorts, when mixed together, at the rate of 5 for 2*d*.,

And consequently $\frac{1}{2}x+\frac{1}{3}x-\frac{4x}{5}=4$, by the question,

That is $15x+10x-24x=120$, or $x=120$, the number of eggs of each sort required.

9. If A can perform a piece of work in 10 days,

and B in 13; in what time will they finish it, if they are both set about it together?

Let the time sought be denoted by x,

Then $\frac{x}{10}=$ the part done by A in one day,

And $\frac{x}{13}=$ the part done by B in one day,

Conseq^y $\frac{x}{10}+\frac{x}{13}=1$ (the whole work) by the quest^n,

That is $13x+10x=130$, or $23x=130$,

Whence $x=\frac{130}{23}=5\frac{15}{23}$ days, the time required.

10. If one agent A, alone, can produce an effect e, in the time a, and another agent B, alone, in the time b; in what time will both of them together produce the same effect?

Let the time sought be denoted by x,

Then $a : e :: x : \frac{ex}{a}=$ part of the effect produced by A,

And $b : e :: x : \frac{ex}{b}=$ part of the effect produced by B,

Hence $\frac{ex}{a}+\frac{ex}{b}=e$ (the whole effect) by the quest.,

Or $\frac{x}{a}+\frac{x}{b}=1$ by dividing by e,

Therefore $x+\frac{ax}{b}=a$, or $bx+ax=ab$,

Consequently $x=\frac{ab}{a+b}=$ time required.

11. How much rye at 4*s*. 6*d*. a bushel, must be mixed with 50 bushels of wheat, at 6*s*. a bushel, so that the mixture may be worth 5*s*. a bushel?

Let $x=$ the number of bushels required,

Then $9x$ is the price of the rye in sixpences,

And 600 the price of the wheat in ditto,

Also $\overline{50+x}\times 10$ the price of the mixture in ditto,

Hence $9x + 600 = 500 + 10x$, by the question,
Or, by transposition, $10x - 9x = 600 - 500$,
Conseq^y^ $x = 100$ the number of bushels required.

12. A labourer engaged to serve for 40 days, on condition that for every day he worked he should receive 20*d.*, but for every day he was absent he should forfeit 8*d.*: now, at the end of the time, he had to receive 1*l.* 11*s.* 8*d.*; how many days then did he work, and how many was he idle?

Let the number of days that he worked be denoted by x,
Then will $40 - x$ be the number of days he was idle,
Also $20x$ the sum earned, and $\overline{40 - x} \times 8$,
Or $320 - 8x$ the sum forfeited,
Hence $20x - (320 - 8x) = 380d. (= 1l.\ 11s.\ 8d.)$,
by the question,
That is $20x - 320 + 8x = 380$,
Or $28x = 380 + 320 = 700$,

Consequently $x = \frac{700}{28} = 25$, the number of days he worked, and $40 - x = 40 - 25 = 15$, the number of days he was idle.

QUESTIONS FOR PRACTICE.

1. It is required to divide a line, of 15 inches in length, into two such parts, that one may be $\frac{3}{4}$ of the other. Ans. $8\frac{4}{7}$ and $6\frac{3}{7}$

2. My purse and money together are worth 12*s.* and 8*d.*; and the money is worth 7 times as much as the purse, how much is there in it?
Ans. 11*s.* 1*d.*

3. A shepherd, being asked how many sheep he had in his flock, said, if I had as many more, half as many more, and 7 sheep and a half, I should have just 500; how many had he? Ans. 197

4. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water; what is its whole length? Ans. 24 feet

5. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left; what then had I at first? Ans. 120 guineas

6. It is required to divide 300*l.* between A, B, and C, so that A may have twice as much as B, and C as much as A and B together. Ans. A 100*l.*, B 50*l.*, C 150*l.*

7. A person, at the time he was married, was 3 times as old as his wife; but after they had lived together 15 years, he was only twice as old; what were their ages on their wedding day? Ans. Bride's age 15, bridegroom's 45

8. What number is that from which, if 5 be subtracted, two thirds of the remainder will be 40. Ans. 65

9. At a certain election, 1296 persons voted, and the successful candidate had a majority of 120; how many voted for each? Ans. 708 for one, and 588 for the other

10. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each? Ans. A's 84, B's 42, and C's 14

11. A person has 6 sons, each of whom is 4

years older than his next younger brother, and the eldest is three times as old as the youngest; what are the ages of each?

Ans. 10, 14, 18, 22, 26, and 30

12. Two persons, A and B, lay out equal sums of money in trade; A gains 126*l.* and B loses 87*l.*, and A's money is now double of B's; what did each lay out? Ans. 300*l.*

13. A person bought a chaise, horse, and harness for 60*l.*; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness; what did he give for each?

Ans. 13*l.* 6*s.* 8*d.* for the horse, 6*l.* 13*s.* 4*d.* for the harness, and 40*l.* for the chaise.

14. A person was desirous of giving 3*d.* a piece to some beggars, but found he had not money enough in his pocket by 8*d.*, he therefore gave them each 2*d.*, and had then 3*d.* remaining; required the number of beggars? Ans. 11

15. A servant agreed to live with his master for 8*l.* a year, and a livery, but was turned away at the end of seven months, and received only 2*l.* 13*s.* 4*d.* and his livery; what was its value? Ans. 4*l.* 16*s.*

16. A person left 560*l.* between his son and daughter, in such a manner, that for every half-crown the son should have, the daughter was to have a shilling; what were their respective shares?

Ans. Son 400*l.*, daughter 160*l.*

17. There is a certain number, consisting of two places of figures, which is equal to four times the sum of its digits; and if 18 be added to it the digits will be inverted; what is the number?

Ans. 24

18. A person being asked how many horses he kept, said, for want of room in my stable, I am obliged to put 8 of them out to hire; but I am now building a new one twice as large as the former, which, when finished, will enable me to accommodate 8 horses more than I now have; how many did he keep? Ans. 24

19. Two persons, A and B, have both the same income; A saves $\frac{1}{5}$ of his yearly, but B, by spending 50*l.* per annum more than A, at the end of four years, finds himself 100*l.* in debt; what was their income? Ans. 125*l.*

20. When a company at a tavern came to pay their reckoning, they found, that if there had been 3 more, they would have had a shilling a piece less to pay, and if there had been two less, they would have had a shilling a piece more to pay; required the number of persons, and the quota of each?

Ans. 12 persons, quota of each 5*s.*

21. What number is that, which being severally added to 36 and 52, will make the former sum to the latter as 3 is to 4? Ans. 12

22. There are two pipes, one of which will fill a cistern in the time a, and the other in the time b; now, if they are both set open together, in what time will they fill it? Ans. $\frac{ab}{a+b}$ time

23. A person at a tavern borrowed as much money as he had about him, and out of the whole spent 1*s.*; he then went to a second tavern, where he also borrowed as much as he had now about him, and out of the whole spent 1*s.*; and going on, in this manner, to a third and fourth

tavern, he found, after spending his shilling at the latter, that he had nothing left; how much money had he at first?

Ans. 11$\frac{1}{4}$*d.*

24. It is required to divide the number 75 into two such parts, that three times the greater may exceed seven times the less by 15.

Ans. 54 and 21

25. In a mixture of wine and cider, $\frac{1}{2}$ of the whole plus 25 gallons was wine, and $\frac{1}{3}$ part minus 5 gallons was cider; how many gallons were there of each? Ans. 85 of wine and 35 of cider

26. A bill of 120*l.* was paid in guineas and moidores, and the number of pieces of both sorts that were used were just 100; how many were there of each, reckoning the moidore at 27*s.*? Ans. 50

27. Two travellers set out at the same time from London and York, whose distance is 197 miles; one of them goes 14 miles a day, and the other 16; in what time will they meet?

Ans. 6 days 13$\frac{3}{5}$ hours

28. There is a fish whose tail weighs 9*lb.*; his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fish?

Ans. 72*lb.*

29. It is required to divide 10 into three such parts, that, if the first be multiplied by 2, the second by 3, and the third by 4, the three products shall be all equal. Ans. 4$\frac{8}{13}$, 3$\frac{1}{13}$, and 2$\frac{4}{13}$

30. It is required to divide the number 36 into

three such parts, that $\frac{1}{2}$ the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third may be all equal to each other.

Ans. The parts are 8, 12, and 16

31. A person has two horses, and a saddle, which, of itself, is worth 50*l.*; now, if the saddle be put on the back of the first horse, it will make his value double that of the second, and if it be put on the back of the second, it will make his value triple that of the first: what is the value of each horse?

Ans. One 30*l.* and the other 40*l.*

32. If A gives B 5*s.* of his money, B will have twice as much as the other has left; and if B gives A 5*s.* of his money, A will have three times as much as the other has left; how much had each?

Ans. A 13*s.* and B 11*s.*

33. What two numbers are those whose difference, sum, and product, are to each other as the numbers 2, 3, and 5 respectively?

Ans. 10 and 2

34. A person in play lost $\frac{1}{4}$ of his money, and then won back 3*s.*, after which he lost $\frac{1}{3}$ of what he now had, and then won back 2*s.*; lastly he lost $\frac{1}{7}$ of what he then had, and after this found he had but 12*s.* remaining; what had he at first?

Ans. 20*s.*

35. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3, but 2 of the greyhound's leaps are as much as three of the hare's; how many leaps must the greyhound take to catch the hare?

Ans. 300

36. It is required to divide the number 90 into four such parts, that if the first part be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient shall be all equal?

Ans. The parts are 18, 22, 10, and 40

37. A man and his wife usually drank out a cask of beer in 12 days, but when the man was from home it lasted the woman 30 days; how many days would the man alone be in drinking it?

Ans. 20 days

38. A general, ranging his army in square battle, finds he has 284 men to spare, but increasing the side by one man, he wants 25 to fill up the square; how many soldiers had he? Ans. 24000

39. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days; how many days will it take each person to perform the same work alone?

Ans. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$

40. A courier sets out from a certain place, and travels at the rate of m miles in n hours; and a hours after, another sets out from the same place, and travels after him at the rate of r miles in s hours; when will the second overtake the first?

Ans. First travels $\frac{anr}{nr-ms}$ hours, second $\frac{ams}{nr-ms}$ hours

41. Given the rates a and b of two ingredients, and the rate c of the compound m; to find what portions x and y of each of them must be taken to compose the mixture.

Ans. $x=m\left(\frac{c-b}{a-b}\right)$, $y=m\left(\frac{a-c}{a-b}\right)$

42. Suppose that 19*lbs.* of gold weigh 18*lbs.* in water, and 10*lbs.* of silver 9*lbs* in water; and that a mass of gold and silver of 106*lbs.* weighs 99*lbs.* in water; it is required to find the quantity of gold and silver contained in the mass?

Ans. 76*lbs.* of gold, and 30*lbs.* of silver

QUADRATIC EQUATIONS.

(P) A QUADRATIC EQUATION, as before observed, is that in which the unknown quantity is of two dimensions, or that rises to the second power; and is either simple or compound.

A simple quadratic equation, is that which contains only the square, or second power, of the unknown quantity, as $ax^2=b$, or $x^2=\frac{b}{a}$; in which case x is $=\sqrt{\frac{b}{a}}$.

A compound quadratic equation, is that which contains both the first and second power of the unknown quantity; as $ax^2+bx=c$, or $x^2+\frac{b}{a}x=\frac{c}{a}$.

In which case, it is to be observed, that every equation of this kind, having any real positive root, will fall under one or other of the three following forms:

1. $x^2+ax=b$ . . . where $x=-\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}+b)}$.

2. $x^2-ax=b$ . . . where $x=+\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}+b)}$.

3. $x^2-ax=-b$ . . . where $x=+\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}-b)}$.

Or, if the second and last terms be taken either positively or negatively, as they may happen to be, the general equation

$$ax^2 \pm bx = \pm c,$$

which comprehends all the three cases above mentioned, may be resolved by means of the following rule;

RULE.

Transpose all the terms that involve the unknown quantity to one side of the equation, and the known terms to the other; observing to arrange them so, that the term which contains the square of the unknown quantity may be positive, and stand first in the equation.

Then, if this square has any coefficient prefixed to it, let all the rest of the terms be divided by it, and the equation will be brought to one of the above forms.

In which state of it, the value of the unknown quantity x, may be readily found; being always equal to half the coefficient, or multiplier of x, in the second term of the equation, taken with a contrary sign, together with *plus* or *minus* the square root of the square of this number, and the known quantity that forms the absolute, or third term, of the equation (*s*).

(*s*) This rule, which is more commodious in its practical application, than that usually given, is founded upon the same principle; being derived from the well known property, that in any quadratic $x^2 \pm ax = \pm b$, if the square of half the coefficient (a) of the second term of the equation be added to each of its sides, so as to render it of the form

$$x^2 \pm ax + \tfrac{1}{4}a^2 = \pm b + \tfrac{1}{4}a^2,$$

that side which contains the unknown quantity will then be a

Note. All equations, which have the index of the unknown quantity, in one of their terms, just double that of the other, are resolved like quadratics, by first finding the value of its square root according to the method used in the above rule, and then taking such a root, or power of the result, as is denoted by the reduced index of the unknown quantity.

Thus, if there be taken any general equation of this kind, as

$$x^{2m} + ax^{m} = b,$$

we shall have, by taking the square root of x^{2m}, observing the latter part of the rule,

$$x^{m} = -\frac{a}{2} \pm \surd\left(\frac{a^2}{4} + b\right), \text{ and } x = \left\{-\frac{a}{2} \pm \surd\left(\frac{a^2}{4} + b\right)\right\}^{\frac{1}{m}}.$$

And if the equation, which is to be resolved, be of the form

$$x^{m} - ax^{\frac{m}{2}} = b,$$

we shall necessarily have, according to the same principle,

$$x^{\frac{m}{2}} = \frac{a}{2} \pm \surd\left(\frac{a^2}{4} + b\right), \text{ and } x = \left\{\frac{a}{2} \pm \surd\left(\frac{a^2}{4} + b\right)\right\}^{\frac{2}{m}}$$

EXAMPLES.

1. Given $x^2 + 4x = 140$, to find the value of x.

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complete square; and, consequently, by extracting its roots, we shall have

$$x \pm \tfrac{1}{2}a = \pm \sqrt{\tfrac{1}{4}a^2 + b}, \text{ or } x = \mp \tfrac{1}{2}a \pm \sqrt{\tfrac{1}{4}a^2 + b},$$

which is the same as the rule.

It may here, also, be observed, that the ambiguous sign ±, which denotes both + and −, is prefixed to the radical part of the value of x in every expression of this kind, because the square root of any positive quantity, as a^2, is either $+a$ or $-a$; for $(+a) \times (+a)$, or $(-a) \times (-a)$ are each $= +a^2$: but the

Here $x^2+4x=140$, by the question,

Whence $x=-2\pm\sqrt{4+140}$, by the rule,

Or, which is the same thing, $x=-2\pm\sqrt{144}$,

Therefore $x=-2+12=10$, or $-2-12=-14$,

Where one of the values of x is positive and the other negative.

2. Given $x^2-12x+30=3$, to find the value of x.

Here $x^2-12x=3-30=-27$, by transposition,

Whence $x=6\pm\sqrt{36-27}$, by the rule,

Or, which is the same thing, $x=6\pm\sqrt{9}$,

Therefore $x=6+3=9$, or $=6-3=3$,

Where it appears that x has two positive values.

3. Given $2x^2+8x-20=70$, to find the value of x.

Here $2x^2+8x=70+20=90$, by transposition,

And $x^2+4x=45$, by dividing by 2,

Whence $x=-2\pm\sqrt{4+45}$, by the rule,

Or, which is the same thing, $x=-2\pm\sqrt{49}$,

Therefore $-2+7=5$, or $=-2-7=-9$,

One of the values of x being positive and the other negative.

4. Given $3x^2-3x+6=5\frac{1}{3}$, to find the value of x.

Here $3x^2-3x=5\frac{1}{3}-6=-\frac{2}{3}$ by transposition,

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square root of a negative quantity, as $-a^2$, is imaginary, or unassignable, there being no quantity, either positive or negative, that, when multiplied by itself, will give a negative product.

To this we also may further add, that from the constant occurrence of the double sign before the radical part of the above expression, it necessarily follows, that every quadratic equation must have two roots; which are either both real, or one real and the other imaginary, according to the nature of the question. See the corresponding article (P) Vol. II.

And $x^2 - x = -\frac{2}{9}$ by dividing by 3,

Whence $x = \frac{1}{2} \pm \sqrt{(\frac{1}{4} - \frac{2}{9})}$, by the rule,

Or, by subtracting $\frac{2}{9}$ from $\frac{1}{4}$, $x = \frac{1}{2} \pm \sqrt{\frac{1}{36}}$,

Therefore $x = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, or $= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$,

In which case x has two positive values.

5. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 20\frac{1}{2} = 42\frac{2}{3}$, to find the value of x.

Here $\frac{1}{2}x^2 - \frac{1}{3}x = 42\frac{2}{3} - 20\frac{1}{2} = 22\frac{1}{6}$ by transposition,

And $x^2 - \frac{2}{3}x = 44\frac{1}{3}$, by dividing by $\frac{1}{2}$, or mult. by 2,

Whence we have $x = \frac{1}{3} \pm \sqrt{(\frac{1}{9} + 44\frac{1}{3})}$, by the rule,

Or, by adding $\frac{1}{9}$ and $44\frac{1}{3}$ together, $x = \frac{1}{3} \pm \sqrt{\frac{400}{9}}$,

Therefore $x = \frac{1}{3} + 6\frac{2}{3} = 7$, or $= \frac{1}{3} - 6\frac{2}{3} = -6\frac{1}{3}$,

Where one value of x is positive, and the other negative.

6. Given $ax^2 + bx = c$, to find the value of x.

Here $x^2 + \frac{b}{a}x = \frac{c}{a}$ by dividing each side by a.

Whence, by the rule, $x = -\frac{b}{2a} \pm \sqrt{(\frac{b^2}{4a^2} + \frac{c}{a})}$,

Or, multip^{g} c and a by $4a$, $x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 + 4ac}{4a^2}}$,

$$\text{Therefore } x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 + 4ac}.$$

7. Given $ax^2 - bx + c = d$, to find the value of x.

Here $ax^2 - bx = d - c$, by transposition,

And $x^2 - \frac{b}{a}x = \frac{d-c}{a}$, by dividing by a.

Whence $x=\frac{b}{2a}\pm\sqrt{(\frac{d-c}{a}+\frac{b^2}{4a^2})}$ by the rule,

Or, mult^s $d-c$ & a by $4a$, $x=\frac{b}{2a}\pm\frac{1}{2a}\sqrt{4a(d-c)+b^2}$.

8. Given $x^4+ax^2=b$, to find the value of x.

Here $x^4+ax^2=b$, by the question,

Or $x^2=-\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}+b)}=-\frac{a}{2}\pm\frac{1}{2}\sqrt{(a^2+4b)}$, by the rule,

Whence $x=\pm\sqrt{(-\frac{a}{2}\pm\frac{1}{2}\sqrt{4b+a^2})}$ by extr^n of roots.

9. Given $\frac{1}{2}x^6-\frac{1}{4}x^3=-\frac{1}{32}$, to find the value of x.

Here $\frac{1}{2}x^6-\frac{1}{4}x^3=-\frac{1}{32}$, by the question,

And $x^6-\frac{1}{2}x^3=-\frac{1}{16}$, by multiplying by 2,

Whence $x^3=\frac{1}{4}\pm\sqrt{(\frac{1}{16}-\frac{1}{16})}=\frac{1}{4}$, by the rule,

And consequently $x=\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{2}{8}}=\frac{1}{2}\sqrt[3]{2}$.

10. Given $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$, to find the value of x.

Here $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$, by the question,

And $x^{\frac{2}{3}}+\frac{3}{2}x^{\frac{1}{3}}=1$, by dividing by 2,

Whence $x^{\frac{1}{3}}=-\frac{3}{4}\pm\sqrt{(\frac{9}{16}+1)}=-\frac{3}{4}\pm\frac{5}{4}=\frac{1}{2}$, or -2.

Therefore $x=(\frac{1}{2})^3=\frac{1}{8}$, or $(-2)^3=-8$.

11. Given $x^4-12x^3+44x^2-48x=9009(a)$, to find the value of x.

This equation may be expressed as follows,

$$(x^2-6x)^2+8(x^2-6x)=a,$$

Whence $x^2-6x=-4\pm\sqrt{16+a}$, by the com. rule,

And, by a second operation, $x=3\pm\sqrt{(9-4\pm\sqrt{16+a})}$

Therefore, by restoring the value of a, we have

$$x = 3 \pm \sqrt{5 \pm \sqrt{9025}},$$

Or, by extraction of roots, $x = 13$, the Ans.

EXAMPLES FOR PRACTICE (*t*).

1. Given $x^2 - 8x + 10 = 19$, to find the value of x.
Ans. $x = 9$

2. Given $x^2 - x - 40 = 170$, to find the value of x.
Ans. $x = 15$

3. Given $3x^2 + 2x - 9 = 76$, to find the value of x.
Ans. $x = 5$

4. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$, to find the value of x.
Ans. $x = 1\frac{1}{2}$

5. Given $\frac{1}{2}x - \frac{1}{3}\sqrt{x} = 22\frac{1}{6}$, to find the value of x.
Ans. $x = 49$

6. Given $x + \sqrt{5x + 10} = 8$, to find the value of x.
Ans. $x = 3$

7. Given $\sqrt{10 + x} - \sqrt[4]{10 + x} = 2$, to find the value of x.
Ans. $x = 6$

8. Given $2x^4 - x^2 + 96 = 99$, to find the value of x.
Ans. $x = \frac{1}{2}\sqrt{6}$

9. Given $x^6 + 20x^3 - 10 = 59$, to find the value of x.
Ans. $x = \sqrt[3]{3}$

(*t*) The unknown quantity in these examples, as well as in those given above, has always two values, as appears from the rule; but the negative roots, being, in general, but seldom used in practical questions of this kind, are here suppressed.

10. Given $3x^{2n}-2x^{n}+3=11$, to find the value of x.
Ans. $x=\sqrt[n]{2}$

11. Given $\frac{2}{3}x\sqrt{3+2x^2}=\frac{1}{2}+\frac{2}{3}x^2$, to find the value of x.
Ans. $x=\frac{1}{2}\sqrt{-3+3\sqrt{2}}$

12. Given $x\sqrt{(\frac{6}{x}-x)}=\frac{1+x^2}{\sqrt{x}}$, to find the value of x.
Ans. $x=(1+\frac{1}{2}\sqrt{2})^{\frac{1}{2}}$

13. Given $\frac{1}{x}\sqrt{1-x^3}=x^2$, to find the value of x.
Ans. $x=(\frac{1}{2}\sqrt{5}-\frac{1}{2})^{\frac{1}{3}}$

14. Given $x\sqrt{(\frac{a}{x}-1)}=\sqrt{x^2-b^2}$, to find the value of x.
Ans. $x=\frac{1}{2}a+\frac{1}{4}\sqrt{8b^2+a^2}$

15. Given $\sqrt{1+x-x^2}-2(1+x-x^2)=\frac{1}{9}$, to find the value of x.
Ans. $x=\frac{1}{2}+\frac{1}{6}\sqrt{33}$

16. Given $\sqrt{(x-\frac{1}{x})}+\sqrt{(1-\frac{1}{x})}=x$, to find the value of x.
Ans. $x=\frac{1}{2}+\frac{1}{2}\sqrt{5}$

17. Given $x^{4n}-2x^{3n}+x^{n}=6$, to find the value of x.
Ans. $x=\sqrt[n]{\frac{1}{2}+\frac{1}{2}\sqrt{13}}$

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

The methods of expressing the conditions of questions of this kind, and the consequent reduction of them, till they are brought to a quadratic equation, involving only one unknown quantity and its square, are the same as those already given for simple equations.

1. To find two numbers such that their difference shall be 8, and their product 240.

Let x equal the least number,
Then will $x+8=$ the greater,
And $x(x+8)=x^2+8x=240$, by the question,
Whence $x=-4+\sqrt{16+240}=-4+\sqrt{256}$, by the rule,
Therefore $x=16-4=12$, the less number,
And $x+8=12+8=20$, the greater.

2. It is required to divide the number 60 into two such parts, that their product may be 864.

Let $x=$ the greater part,
Then will $60-x=$ the less,
And $x(60-x)=60x-x^2=864$, by the question,
Or $x^2-60x=-864$, by changing the signs,
Whence $x=30\pm\sqrt{900-864}=30\pm\sqrt{36}=30\pm6$, by the rule,
And consequently $x=30+6=36$, or $30-6=24$, the two parts.

3. It is required to find two numbers such that their sum shall be 10(a), and the sum of their squares 58(b).

Let $x=$ the greater of the two numbers,
Then will $a-x=$ the less,
And $x^2+(a-x)^2=2x^2-2ax+a^2=b$, by the question,
Or $2x^2-2ax=b-a^2$, by transposition,
And $x^2-ax=\frac{b-a^2}{2}$, by division,
Whence $x=\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}+\frac{b-a^2}{2})}=\frac{a}{2}\pm\frac{1}{2}\sqrt{2b-a^2}$ by the rule,
And if 10 be put for a, and 58 for b, we shall have
$x=\frac{10}{2}+\frac{1}{2}\sqrt{116-100}=7$, the greater number,
And $10-x=\frac{10}{2}-\frac{1}{2}\sqrt{116-100}=3$, the less.

4. Having sold a piece of cloth for 24*l.*, I gained as much per cent. as it cost me; what then was the price of the cloth?

Let $x=$ pounds the cloth cost,

Then will $24-x=$ the whole gain,

But $100 : x :: x : 24-x$, by the question,

Or $x^2=100(24-x)=2400-100x$,

That is, $x^2+100x=2400$,

Whence $x=-50+\sqrt{2500+2400}=-50+70=20$, by the rule,

And consequently 20*l.* = price of the cloth.

5. A person bought a number of sheep for 80*l.*, and if he had bought 4 more for the same money, he would have paid 1*l.* less for each; how many did he buy?

Let x represent the number of sheep,

Then will $\frac{80}{x}$ be the price of each,

And $\frac{80}{x+4}=$ price of each, if $x+4$ cost 80*l.*

But $\frac{80}{x}=\frac{80}{x+4}+1$, by the question,

Or $80=\frac{80x}{x+4}+x$, by multiplication,

And $80x+320=80x+x^2+4x$, by the same,

Or, by leaving out $80x$ on each side, $x^2+4x=320$,

Whence $x=-2+\sqrt{4+320}=-2+18$, by the rule,

And consequently $x=16$, the number of sheep.

6. It is required to find two numbers, such that their sum, product, and difference of their squares, shall be all equal.

Let $x=$ the greater number, and $y=$ the less.

Then $\left\{\begin{array}{l} x+y=xy \\ x+y=x^2-y^2 \end{array}\right\}$ by the question, Hence $1 = \frac{x^2-y^2}{x+y} = x - y$, or $x = y + 1$, by 2d equation,

And $(y+1) + y = y(y+1)$ by 1st equation,

That is, $2y + 1 = y^2 + y$; or $y^2 - y = 1$,

Whence $y = \frac{1}{2} + \sqrt{(\frac{1}{4} + 1)} = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, by the rule,

Therefore $y = \frac{1}{2} + \frac{1}{2}\sqrt{5} = 1.6180$&c.

And $x = y + 1 = \frac{3}{2} + \frac{1}{2}\sqrt{5} = 2.6180$&c.

7. It is required to find four numbers in arithmetical progression, such that the product of the two extremes shall be 45, and the product of the means 77.

Let $x =$ least extreme, and $y =$ common difference,
Then x, $x+y$, $x+2y$, and $x+3y$, will be the four numbers,

$$\text{Hence}\left\{\begin{array}{l} x(x+3y) = x^2 + 3xy = 45 \\ \overline{x+y}(x+2y) = x^2 + 3xy + 2y^2 = 77 \end{array}\right\}\text{by the quest}^n,$$

And $2y^2 = 77 - 45 = 32$, by subtraction,

Or $y^2 = \frac{32}{2} = 16$ by division, and $y = \sqrt{16} = 4$,

Therefore $x^2 + 3xy = x^2 + 12x = 45$, by the 1st equat^n,

And consequently $x = -6 + \sqrt{\overline{36+45}}, = -6 + 9 = 3$, by the rule,

Whence the numbers are 3, 7, 11, and 15.

8. It is required to find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84.

Let x, y, and z be the three numbers,

Then $xz = y^2$, by the nature of proportion,

$$\text{And}\left\{\begin{array}{l} x + y + z = 14 \\ x^2 + y^2 + z^2 = 84 \end{array}\right\}\text{by the question,}$$

Hence $x + z = 14 - y$, by the second equation, And $x^2+2xz+z^2=196-28y+y^2$, by squaring both sides,

Or $x^2+z^2+2y^2=196-28+y^2$ by putting $2y^2$ for its equal $2xz$,

That is $x^2+y^2+z^2=196-28y$ by subtraction,

Or $196-28y=84$ by equality,

Hence $y=\frac{196-84}{28}=4$, by transposition and div^n,

Again $xz=y^2=16$, or $x=\frac{16}{z}$, by the 1st equation,

And $x+y+z=\frac{16}{z}+4+z=14$, by the 2d equation,

Or $16+4z+z^2=14z$, or $z^2-10z=-16$,

Whence $z=5\pm\sqrt{25-16}=5\pm3=8$, or 3 by the rule,

Therefore the three numbers are 2, 4, and 8.

9. It is required to find two numbers, such that their sum shall be 13(a), and the sum of their fourth powers 4721(b).

Let $x=$ the difference of the two numbers sought,

Then will $\frac{1}{2}a+\frac{1}{2}x$, or $\frac{a+x}{2}=$ the greater number,

And $\frac{1}{2}a-\frac{1}{2}x$, or $\frac{a-x}{2}=$ the less,

But $\frac{(a+x)^4}{16}+\frac{(a-x)^4}{16}=b$, by the question,

Or $(a+x)^4+(a-x)^4=16b$, by multiplication,

Or $2a^4+12a^2x^2+2x^4=16b$, by involution and add^n,

And $x^4+6a^2x^2=8b-a^4$ by transpos^n and division,

Whence $x^2=-3a^2+\sqrt{9a^4+8b-a^4}=-3a^2+\sqrt{8(a^4+b)}$, by the rule,

And $x=\sqrt{-3a^2+2\sqrt{2(a^4+b)}}$, by extracting the root,

Where, by substituting 13 for a, and 4721 for b, we shall have $x=3$,

Therefore $\frac{13+x}{2}=\frac{16}{2}=8$, the greater number, And $\frac{13-x}{2}=\frac{10}{2}=5$, the less number,

10. Given the sum of two numbers equal to s, and their product equal to p, to find the sum of their squares, cubes, biquadrates, &c.

Let x and y denote the two numbers,

Then will $x+y=s$, and $xy=p$, by the question.

From the square of the first equation take twice the second, and we shall have

$x^2+y^2=s^2-2p$, the sum of their squares.

Multiply this by the 1st equation, and the product will be $x^3+xy^2+x^2y+y^3=s^3-2ps$.

From which subtract the product of the first and second equations, and there will remain

$x^3+y^3=s^3-3sp$, the sum of the cubes.

Multiply this likewise by the 1st equation, and the product will be $x^4+xy^3+x^3y+y^4=s^4-3s^2p$. From which subtract the product of the second and third, and there will remain

$x^4+y^4=s^4-4s^2p+2p^2$, the sum of the biquadrates.

And, by proceeding to multiply the sum of the powers last found by the first equation, and subtracting from the result the product of the two next preceding equations, we shall obtain the sum of any powers proposed: thus,

$$x^5+y^5=s^5-5s^3p+5sp^2,$$

the sum of the fifth powers, or, by expressing the theorem in general terms,

$$x^m+y^m=s^m-ms^{m-2}p+\frac{m(m-3)}{2}s^{m-4}p^2-\frac{m(m-4)(m-5)}{2\,.\,3}$$
$$s^{m-6}p^3+\frac{m(m-5)(m-6)(m-7)}{2\,.\,3\,.\,4}s^{m-8}p^4-\ \&\text{c.}$$

Where it is evident that the series, or expression for the sum of the powers, will terminate when the least index of s becomes $=0$.

QUESTIONS FOR PRACTICE.

1. It is required to divide the number 40 into two such parts, that the sum of their squares shall be 818. Ans. 23 and 17

2. To find a number such, that if you subtract it from 10, and then multiply the remainder by the number itself, the product shall be 21.
Ans. 7 or 3

3. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference. Ans. 10 and 14

4. It is required to divide a line, of 20 inches in length, into two such parts, that the rectangle of the whole and one of the parts shall be equal to the square of the other. Ans. $10\sqrt{5}-10$

5. It is required to divide the number 60 into two such parts, that their product shall be to the sum of their squares in the ratio of 2 to 5.
Ans. 20 and 40

6. It is required to divide 146 into two such parts, that the difference of their square roots may be 6. Ans. 25 and 121

7. What two numbers are those whose sum is 20 and their product 36? Ans. 2 and 18

8. The sum of two numbers is $1\frac{1}{3}$, and the sum of their reciprocals $3\frac{1}{5}$; required the numbers.
Ans. $\frac{1}{2}$ and $\frac{5}{6}$

9. The difference of two numbers is 15, and half their product is equal to the cube of the less number; required the numbers. Ans. 3 and 18

10. The difference of two numbers is 5, and the

difference of their cubes 1685; required the numbers. Ans. 8 and 13

11. A person bought cloth for 33*l.* 15*s.*, which he sold again at 2*l.* 8*s.* per piece, and gained by the bargain as much as one piece cost him; required the number of pieces. Ans. 15

12. What two numbers are those, whose sum, multiplied by the greater, is equal to 77, and whose difference, multiplied by the less, is equal to 12. Ans. 4 and 7

13. A grazier bought as many sheep as cost him 60*l.*, and after reserving 15 out of the number, sold the remainder for 54*l.*, and gained 2*s.* a head by them: how many sheep did he buy? Ans. 75

14. It is required to find two numbers, such that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes. Ans. $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5+\sqrt{5})$

15. The difference of two numbers is 8, and the difference of their fourth powers is 14560; required the numbers. Ans. 3 and 11

16. What number is that, the quotient of which, when divided by the product of its two digits, shall be 2; and if 27 be added to it, the digits will be inverted? Ans. 36

17. The sum of three numbers in harmonical proportion is 191, and the product of the first and third is 4032; required the numbers.

Ans. 72, 63, and 56

18. A company at a tavern had 8*l.* 15*s.* to pay for their reckoning; but, before the bill was settled, two of them went away; in consequence

of which those who remained had 10*s*. a piece more to pay than before: how many were there in company? Ans. 75

19. A person ordered 7*l*. 4*s*. to be distributed among some poor people; but, before the money was divided, there came in, unexpectedly, two claimants more, by which means the former received a shilling a piece less than they would otherwise have done; what was their number at first? Ans. 16 persons

20. The sum of four numbers in arithmetical progression is 56, and the sum of their squares is 864; what are the numbers?

Ans. 8, 12, 16, and 20

21. It is required to find four numbers in geometrical progression such, that their sum shall be 15, and the sum of their squares 85.

Ans. 1, 2, 4, and 8

22. To find two numbers such, that their product shall be 320, and the difference of their cubes be to the cube of their difference in the ratio of 61 to 1.

Ans. 20 and 16

23. The sum of two numbers is 11, and the sum of their fifth powers is 17831; required the numbers? Ans. 4 and 7

24. It is required to find four numbers in arithmetical progression such, that their common difference shall be 4, and their continued product 176985. Ans. 15, 19, 23, and 27

25. Two detachments of foot being ordered to a station at the distance of 39 miles from their present quarters, begin their march at the same time; but one party, by travelling $\frac{1}{4}$ of a mile an hour

faster than the other, arrive there an hour sooner; required their rates of marching?

Ans. $3\frac{1}{4}$ and 3 miles per hour

26. It is required to find two numbers, such that the square of the first plus their product, shall be 140, and the square of the second minus their product 78. Ans. 7 and 13

Of the solution of quadratic equations, by the tables of sines and tangents.

When the numeral parts of a quadratic equation are either large numbers, or complicated fractions, its solution may be more readily obtained by means of one of the following trigonometrical formulæ, than by the common method, which, in this case, is often very laborious.

1. $x^2+ax=b,$

$$\text{Put } \frac{2r}{a}\sqrt{b}=\text{tan. } z,$$

$$\text{Then } x=\begin{cases}+\frac{1}{r}\sqrt{b}\times\text{ tan. }\frac{1}{2}z,\\-\frac{1}{r}\sqrt{b}\times\text{ cot. }\frac{1}{2}z.\end{cases}$$

2. $x^2-ax=b,$

$$\text{Put } \frac{2r}{a}\sqrt{b}=\text{tan. } z.$$

$$\text{Then } x=\begin{cases}-\frac{1}{r}\sqrt{b}\times\text{ tan. }\frac{1}{2}z,\\+\frac{1}{r}\sqrt{b}\times\text{ cot. }\frac{1}{2}z.\end{cases}$$

3. $x^2-ax=-b,$

$$\text{Put } \frac{2r}{a}\sqrt{b}=\text{sin. } z,$$

$$\text{Then } x=\begin{cases}+\frac{1}{r}\sqrt{b}\times\text{ tan. }\frac{1}{2}z,\\+\frac{1}{r}\sqrt{b}\times\text{ cot. }\frac{1}{2}z.\end{cases}$$

In the last of which cases, it is to be observed, that if $\frac{2r}{a}\sqrt{b}$ be greater than the radius r, or $4b$ greater than a^2, the two roots, or values of x, will be impossible (u).

EXAMPLES.

1. Given $x^2+\frac{7}{44}x=\frac{1695}{12716}$, to find the two roots of the equation.

$$\text{Here tan. } z=\frac{88r}{7}\sqrt{\frac{1695}{12716}}$$

Log. 1695 . . .	3.2291697
Log. 12716 . .	4.1043505
	2)$\bar{1}$.1248192
Half sum . . .	$\bar{1}$.5624096
Log. 88 . . .	1.9444827
Colog. 7 . . .	9.1549020
Log. tan. z . .	10.6617943

Or $z=77^\circ\ 42'\ 32''$, and $\frac{1}{2}z=38^\circ\ 51'\ 16''$, whence

Log. tan. $\frac{1}{2}z$ . .	9.9061115
Former $\frac{1}{2}$ sum . .	$\bar{1}$.5624096
Log. $\frac{1}{r}$ . . .	$\overline{10}$.0000000
Log. x	$\bar{1}$.4685211

Or $x=.2941176=\frac{5}{17}$, the positive root.

(u) When the absolute term b is small with respect to the coefficient of x, the roots, or values of x, in the equation $x^2\pm ax=\pm b$, may also be determined by means of the following series.

And if cot. $\frac{1}{2}x$ be taken, instead of tan. $\frac{1}{2}x$, the other value of x will be found $= -.4532085$,

Or $-\frac{1695}{12716} + \frac{5}{17} = -\frac{339}{748}$, the negative root.

2. Given $x^2 + 1728\frac{1}{3}x = 123578$, to find the two roots, or values of x. Ans. $x = \begin{cases} +4729.6619 \\ -6457.9953 \end{cases}$

3. Given $x^2 - \frac{127}{364}x = \frac{2547}{59825}$, to find the two roots, or values of x. Ans. $x = \begin{cases} +.430913 \\ -.109483 \end{cases}$

4. Given $x^2 - \frac{101}{102}x = -\frac{19003}{119072}$, to find the two roots, or values of x. Ans. $x = \begin{cases} +.522327 \\ +.305541 \end{cases}$

OF CUBIC EQUATIONS.

(a) A cubic equation is that in which the unknown quantity rises to three dimensions; and like quadratics, or those of the higher orders, is either simple or compound.

A simple cubic equation is of the form $ax^3 = b$, or $x^3 = \frac{b}{a}$, where the value of x is $= \sqrt[3]{\frac{b}{a}}$

A compound cubic equation is of the form $x^3 + ax = b$, $x^3 + ax^2 = b$, or $x^3 + ax^2 + bx = c$; in each of which, the known quantities, a, b, c, may be either + or −.

Or, the two latter of these equations may be re-

$$x = \mp a \pm \left(\frac{a}{2} \pm \frac{b}{a} - \frac{b^2}{a^3} \pm \frac{2b^3}{a^5} - \frac{20b^5}{a^7}, \&c.\right);$$

but as the operation employed for extracting the square root of a quantity, is nearly as easy as that of division, this method, in the present instance, possesses no advantage over that in common use.

duced to the same form as the first, by taking away their second terms; which is done as follows:

RULE.

Take some new unknown quantity, and add it to a third part of the coefficient of the second term of the equation with its sign changed; then if this sum, or difference, as it may happen to be, be substituted for the original unknown quantity and its powers, in the proposed equation, there will arise an equation wanting its second term (x).

Note. The second term of any of the higher orders of equations may also be exterminated in a similar manner, by substituting for the unknown quantity the sum of some other unknown quantity, and the 4th, 5th, &c. part of the coefficient of its second term, with the sign changed, according as the equation is of the 4th, 5th, &c. power.

(x) If, instead of the regular reduction of the equation

$$x^3+ax^2+bx+c=0$$

to another, wanting the second term, as pointed out in the rule, there be put, $x=\frac{1}{3}(y-a)$, we shall have

$$y^3+(9b-3a^2)y=9ab-27c-2a^3;$$

where, since the value of y can be determined, by either of the formulæ hereafter given for this purpose, the value of x will also be known.

And if $b=0$, or the original equation be of the form

$$x^3+ax^2+c=0,$$

the reduced equation will be $y^3-3a^2y+2a^3+27c=0$, or

$$y^3-3a^2y=-2a^3-27c,$$

where the value of y, being found as above, we shall have $x=\frac{1}{3}(y-a)$

EXAMPLES.

1. It is required to exterminate the second term of the equation $x^3+3ax^2=b$, or $x^3+3ax^2-b=0$.

$$\text{Here } x=z-\frac{3a}{3}=z-a,$$

$$\text{Then}\left\{\begin{array}{rl} x^3 = & z^3-3az^2+3a^2z-a^3 \\ 3ax^2 = & \quad\;\, +3az^2-6a^2z+3a^3 \\ -b = & \qquad\qquad\qquad\quad\;\; -b \end{array}\right.$$

$$\text{Whence } z^3-3a^2z+2a^3-b=0,$$

$$\text{Or } z^3-3a^2z=b-2a^3,$$

an equation in which the second power (z^2), of the unknown quantity, is wanting.

2. Let the equation $x^3-12x^2+3x=-16$, be transformed into another, that shall want the second term.

$$\text{Here } x=z+4,$$

$$\text{Then}\left\{\begin{array}{rl} (z+4)^3 = & z^3+12z^2+48z+64 \\ -12(z+4)^2 = & \quad\;\; -12z^2-96z-192 \\ +3(z+4) = & \qquad\qquad\quad +3z\;\;+12 \end{array}\right.$$

$$\text{Hence } z^3-45z-116=-16$$

$$\text{Or } z^3-45z=100$$

an equation where z^2, or the second term, is wanting, as before.

3. Let the equation $x^3-6x^2=10$, be transformed into another, that shall want the second term.

Ans. $y^3-12y=26$

4. Let $y^3-15y^2+81y=243$, be transformed into an equation that shall want the second term.

Ans. $x^3+6x=88$

5. Let the equation $x^3+\frac{3}{4}x^2+\frac{7}{8}x-\frac{9}{16}=0$, be transformed into another, that shall want the second term.

Ans. $y^3+\frac{11}{16}y=\frac{3}{4}$

Of the solution of cubic equations.

RULE.

Take away the second term of the equation, when necessary, as directed in the preceding rule.

Then, if the numbers in the given equation be substituted for a and b in the formula given below, to which it is found to belong, the result will give one of the roots, as required.

1. $x^3 + ax = b$

$$x = \left\{ \begin{array}{l} \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} \\ \text{or} \\ \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} - \dfrac{\frac{1}{3}a}{\sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}}} \end{array} \right.$$

2. $x^3 - ax = b$

$$x = \left\{ \begin{array}{l} \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} + \sqrt[3]{\left(\frac{b}{2}\right) - \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} \\ \text{or} \\ \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} + \dfrac{\frac{1}{3}a}{\sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}}} \end{array} \right.$$

$x^3 - ax = -b$

$$x = \left\{ \begin{array}{l} \sqrt[3]{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} - \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} \\ \text{or} \\ \sqrt[3]{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} + \dfrac{\frac{1}{3}a}{\sqrt[3]{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}}} \end{array} \right.$$

Where it may be observed, that when $(\frac{a}{3})^3$, in the second and third forms, is greater than $(\frac{b}{2})^2$, or $4a^3$ greater than $27b^2$, the solution of the equation, by these rules, is impossible; the question, in this instance, falling under what is usually called the irreducible case of cubic equations (*y*).

EXAMPLES.

1. Given $2x^3-12x^2+36x=44$, to find the value of x.

Here $x^3-6x^2+18x=22$, by dividing by 2.

And, in order to exterminate the second term,

$$\text{Put } x=z+\frac{6}{3}=z+2,$$

$$\text{Then}\left|\begin{array}{rl}(z+2)^3= & z^3+6z^2+12z+8\\ -6(z+2)^2= & \quad\;\; -6z^2-24z-24\\ 18(z+2)\;= & \qquad\qquad\quad 18z+36\end{array}\right|=22$$

$$\text{Whence } z^3+6z+20=22$$

$$\text{Or } z^3+6z=2,$$

And, consequently, by substituting 6 for a, and 2 for b, in the first formula, we shall have,

$$x=\sqrt[3]{\{\frac{2}{2}+\sqrt{(\frac{4}{4}+\frac{216}{27})}\}}+\sqrt[3]{\{\frac{2}{2}-\sqrt{(\frac{4}{4}+\frac{216}{27})}\}}=$$

$$\sqrt[3]{1+\sqrt{(1+8)}}+\sqrt[3]{1-\sqrt{(1+8)}}=\sqrt[3]{1+\sqrt{9}}+$$

$$\sqrt[3]{1-\sqrt{9}}=\sqrt[3]{1+3}+\sqrt[3]{1-3}=\sqrt[3]{4}-\sqrt[3]{2},$$

(*y*) It may here be observed, as a remarkable circumstance in the history of this science, that the solution of the case above mentioned, except by means of a table of sines, or by infinite series, has hitherto resisted the united efforts of the most celebrated mathematicians in Europe, although it is well known that all the three roots of the equation are, in this case, real; whereas, Therefore $x = z + 2 = \sqrt[3]{4} - \sqrt[3]{2} + 2 = 2 + 1.587401 - 1.259921 = 2.32748$, the answer.

2. Given $x^3 - 6x = 12$, to find the value of x.

Here a being equal to 6, and b equal to 12, we shall have, by the second formula,

$$x = \sqrt[3]{6 + \sqrt{(36-8)}} + \frac{2}{\sqrt[3]{\{6 + \sqrt{(36-8)}\}}} =$$

$$\sqrt[3]{6 + \sqrt{28}} + \frac{2}{\sqrt[3]{(6 + \sqrt{28})}} = \sqrt[3]{(6 + 5.2915)} +$$

$$\frac{2}{\sqrt[3]{(6 + 5.2915)}} = \sqrt[3]{11.2915} + \frac{2}{\sqrt[3]{(11.2915)}} = 2.2435 +$$

$$\frac{2}{2.2435} = 2.2435 + .8957 = 3.1392$$

Therefore $x = 3.1392$, the answer.

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in those that are resolvable by the above formulæ, only one of the roots is real. See corresponding Art. Vol. II.

Certain equations, of all orders, having a particular relation of their coefficients, may also be resolved by formulæ similar to those above given for cubic equations.

Thus, in any equation of the following general form,

$$x^n + ax^{n-2} + \frac{n-3}{2n}a^2x^{n-4} + \frac{(n-4)(n-5)}{2.3.n^2}a^3x^{n-6} +$$
$$\frac{(n-5)(n-6)(n-7)}{2.3.4.n^3}a^4x^{n-8}\&c. = b$$

one of the roots, or values of the unknown quantity, will be

$$x = \sqrt[n]{\frac{b}{2} + \sqrt{\left\{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{n}\right)^n\right\}}} - \frac{\frac{1}{n}a}{\sqrt[n]{\frac{b}{2} + \sqrt{\left\{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{n}\right)^n\right\}}}}$$

If, for instance, $x^5 + ax^3 + \frac{1}{5}a^2x = b$, we shall have, by means of this formula,

$$x = \sqrt[5]{\frac{b}{2} + \sqrt{\left\{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{5}\right)^5\right\}}} - \frac{\frac{1}{5}a}{\sqrt[5]{\frac{b}{2} + \sqrt{\left\{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{5}\right)^5\right\}}}}$$

But these equations seldom occur in the resolution of problems.

3. Given $x^3-2x=-4$, to find the value of x.

Here a being $=2$, and $b=4$, we shall have, by the third formula,

$$x=\sqrt[3]{\left\{-2+\sqrt{\left(4-\frac{8}{27}\right)}\right\}}-\sqrt[3]{\left\{2+\sqrt{\left(4-\frac{8}{27}\right)}\right\}}=$$

$$\sqrt[3]{\left(-2+\sqrt{\frac{100}{27}}\right)}-\sqrt[3]{\left(2+\sqrt{\frac{100}{27}}\right)}=\sqrt[3]{\left(-2+\frac{10}{3}\sqrt{\frac{1}{3}}\right)}-$$

$$\sqrt[3]{\left(2+\frac{10}{3}\sqrt{\frac{1}{3}}\right)}=\sqrt[3]{\left(-2+\frac{10}{9}\sqrt{3}\right)}-\sqrt[3]{\left(2+\frac{10}{9}\sqrt{3}\right)}=$$

$$\sqrt[3]{\left\{-2+\frac{10}{9}(1.73205)\right\}}-\sqrt[3]{\left\{2+\frac{10}{9}(1.73205)\right\}}=$$

$$\sqrt[3]{-2+1.9245}-\sqrt[3]{2+1.9245}=\sqrt[3]{-.0755}-$$

$$\sqrt[3]{3.9245}=-.4226-1.5773=-2$$

Therefore $x=-2$, the answer (z).

Note. The other two roots of the equations given above, which are omitted in the answers, may be determined as follows:

Let the root, found by the common formula, be denoted by r, and put all the terms of the equation, when brought to the left hand side, $=0$; then if the equation, so formed, be divided by $x\mp r$, according as r is positive or negative, there will arise a quadratic equation, the roots of which will be the other two roots of the given cubic equation.

Or if, instead of the operation here mentioned, there be put

(z) When the root of the given equation is a whole number, this method only determines it by an approximation of 9s in the decimal part, which sufficiently indicates the entire integer; but in most instances of this kind, its value may be more readily found, by a few trials, from the equation itself.

Or if, as in the above example, the values of

$$\alpha = \sqrt[3]{\left\{ \pm \frac{1}{2}b + \sqrt{(\frac{1}{2}b)^2 \pm (\frac{1}{3}a)^3} \right\}}$$

$$\beta = \sqrt[3]{\left\{ \pm \frac{1}{2}b - \sqrt{(\frac{1}{2}b)^2 \pm (\frac{1}{3}a)^3} \right\}}$$

the three roots of any cubic equation, of the following general form, may be exhibited as below:

$$x^3 \pm ax = \pm b,$$

$\alpha + \beta =$ first root, as found by the rule,

$-\frac{1}{2}(\alpha + \beta) + \frac{1}{2}(a - \beta)\sqrt{-3} =$ second root,

$-\frac{1}{2}(\alpha + \beta) - \frac{1}{2}(\alpha - \beta)\sqrt{-3} =$ third root,

Where it may be observed, that although the two last of these roots, in their general form, have an imaginary appearance, yet, by substituting particular numbers for a and b, they will come out in a real form, in all cases where they ought to do so (*a*).

4. Given $x^3 - 15x = 4$, to find the three roots, or values of x.

$$-\sqrt[3]{(2 + \frac{10}{9}\sqrt{3})} \text{ and } -\sqrt[3]{(2 - \frac{10}{9}\sqrt{3})}$$

be determined according to the method pointed out in the note to Art. (I), Case II, the result will be found equal to -2, as it ought.

(*a*) In the first rule, given in the above Note, for finding all the three roots of the equation

$$x^3 - ax = b,$$

when it falls under the irreducible case, it may be observed, that if r be made to denote the first root, or that found by the common formula, the other two roots, or values of x, will be

$$x = \mp \frac{r}{2} + \frac{1}{2}\sqrt{4a - 3r^2}, \quad x = \mp \frac{r}{2} - \frac{1}{2}\sqrt{4a - 3r^2},$$

where $-\frac{r}{2}$ is to be taken when r is positive, and $+\frac{r}{2}$ when it is negative.

The same rule is also equally applicable to the equation

$$x^3 = 1, \text{ or } x^3 - 1 = 0,$$

Here x is readily found, by a few trials, to be equal to 4, and therefore

$$\begin{array}{l}
x-4)x^3-15x-4(x^2+4x+1\\
\quad\;\; x^3-4x^2\\
\quad\;\; \overline{\quad\quad\quad\quad}\\
\quad\quad\quad 4x^2-15x\\
\quad\quad\quad 4x^2-16x\\
\quad\quad\quad \overline{\quad\quad\quad\quad\quad}\\
\quad\quad\quad\quad\quad\quad x-4\\
\quad\quad\quad\quad\quad\quad x-4\\
\quad\quad\quad\quad\quad\quad \overline{\quad\quad}\\
\quad\quad\quad\quad\quad\quad\quad *
\end{array}$$

Whence, according to the first part of the rule,

$$x^2+4x+1=0, \text{ or } x^2+4x=-1;$$

the two roots of which quadratic are $-2+\sqrt{3}$ and $-2-\sqrt{3}$; and consequently 4, $-2+\sqrt{3}$, and $-2-\sqrt{3}$ are the three roots of the given equation.

Or, putting $a=15$ and $b=4$, we shall have

$$\alpha=\sqrt[3]{\frac{b}{2}+\sqrt{(\frac{b^2}{4}-\frac{a^3}{27})}}=\sqrt[3]{2+\sqrt{(4-125)}}=$$

$$\sqrt[3]{2+\sqrt{-121}}=2+\sqrt{-1}, \text{ and}$$

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which, in conformity to the general principle before mentioned, will be found, like other cubics, to have three roots.

For one of these roots being obviously 1, we shall have, as above,

$$\begin{array}{l}
x-1)x^3-1(x^2+x+1\\
\quad\;\; x^3-x^2\\
\quad\;\; \overline{\quad\quad\quad}\\
\quad\quad\quad x^2-1\\
\quad\quad\quad x^2-x\\
\quad\quad\quad \overline{\quad\quad\quad}\\
\quad\quad\quad\quad x-1\\
\quad\quad\quad\quad x-1\\
\quad\quad\quad\quad \overline{\quad\quad}\\
\quad\quad\quad\quad\quad *
\end{array}$$

Whence, since x^3-1 and $x-1$ are each $=0$, it is plain, that

$$x^2+x+1=0; \text{ or } x^2+x=-1.$$

$$\beta = \sqrt[3]{\frac{b}{2} - \sqrt{(\frac{b^2}{4} - \frac{a^3}{27})}} = \sqrt[3]{2 - \sqrt{(4 - 125)}} =$$

$$\sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-1},$$

as will be found either by cubing $2 + \sqrt{-1}$ and $2 - \sqrt{-1}$, or by the rule given in the note to Art (1), Case II.

Whence $\alpha + \beta = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$,

$-\frac{1}{2}(\alpha + \beta) + \frac{1}{2}(\alpha - \beta)\sqrt{-3} = -2 + \sqrt{-1} \times \sqrt{-3} = -2 + \sqrt{3}$,

$-\frac{1}{2}(\alpha + \beta) - \frac{1}{2}(\alpha - \beta)\sqrt{-3} = -2 - \sqrt{-1} \times \sqrt{-3} = -2 - \sqrt{3}$;

and consequently 4, $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$ are the three roots of the equation, as before found.

5. Given $x^3 + 3x^2 - 6x = 8$, to find the root of the equation, or the value of x. Ans. $x = 2$

6. Given $x^3 + x^2 = 500$, to find the root of the equation, or the value of x. Ans. $x = 7.617$

7. Given $x^3 - 48x^2 = -200$, to find the root of the equation, or the value of x.
Ans. $x = 47.9128$

8. Given $x^3 - 6x = 6$, to find the root of the equation, or the value of x. Ans. $x = \sqrt[3]{4} + \sqrt[3]{2}$

9. Given $x^3 + 9x = 6$, to find the root of the equation, or the value of x. Ans. $x = \sqrt[3]{9} - \sqrt[3]{3}$

And as the two roots of this last equation are

$$-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3} \text{ and } -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3},$$

it follows, that the three roots of the equation $x^3 - 1 = 0$, or, which is the same thing, the three cube roots of 1 are

$$1, \ -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \text{ and } -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

And in the same manner it will be found, that the roots of the equation $x^3 + 1 = 0$, or the three cube roots of -1, are

$$-1, \ +\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \text{ and } -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

10. Given $x^3 - 22x = 24$, to find the root of the equation, or the value of x.

Ans. $x = 5.162277$

11. Given $x^3 - 17x^2 + 54x = 350$, to find the root of the equation, or the value of x.

Ans. $x = 14.954068$

12. Given $x^3 - 6x = 4$, to find the three roots of the equation, or the three values of x.

Ans. -2, $1 + \sqrt{3}$, and $1 - \sqrt{3}$

13. Given $x^3 - 9x = 5$, to find the three roots of the equation, or the three values of x.

Ans. 3.246550, −2.669664, and −.576886

Of the solution of cubic equations, by the tables of sines and tangents.

Cubic equations may often be more readily resolved by means of the following trigonometrical formulæ, than by the method before given; particularly when they fall under what is usually called the irreducible case, in which the common method is known to fail.

$$1.\quad x^3 + ax = \pm b,$$

$$\text{Put } \frac{2ar}{3b}\sqrt{\frac{a}{3}} = \tan.\, z, \text{ and } \sqrt[3]{r^2 \tan.\tfrac{1}{2}z} = \tan.\, u,$$

$$\text{Then } x = \pm\frac{2}{r}\sqrt{\frac{a}{3}} \times \cot.\, 2u$$

Where the upper or under sign is to be taken according as b is positive or negative.

$$2.\quad x^3 - ax = \pm b.$$

This form has two cases, according as $\frac{2ar}{3b}\sqrt{\frac{a}{3}}$ less or greater than radius r, or $4a^3$ less or greater than $27b^2$.

In the first of these cases, or when $4a^3 \sqsubset 27b^2$,

$$\text{Put } \frac{2ar}{3b}\sqrt{\frac{a}{3}} = \sin. z, \text{ and } \sqrt[3]{r^2 \tan. \tfrac{1}{2}z} = \tan. u,$$

$$\text{Then } x = \pm\frac{2}{r}\sqrt{\frac{a}{3}} \times \operatorname{cosec.} 2u$$

Where the upper or under sign is to be taken as above.

In the second case, or when $4a^3 \sqsupset 27b^2$,

$$\text{Put } \frac{3br}{2a}\sqrt{\frac{3}{a}} = \cos. z,$$

Then x will have the three following values:

$$x = \pm\frac{2}{r}\sqrt{\frac{a}{3}} \times \cos. \frac{z}{3},$$

$$x = \mp\frac{2}{r}\sqrt{\frac{a}{3}} \times \cos. (60^\circ - z)$$

$$x = \mp\frac{2}{r}\sqrt{\frac{a}{3}} \times \cos. (60^\circ + z)$$

Where the upper sign is to be used when b is positive, and the under sign when it is negative, as in the other cases.

EXAMPLES.

1. Given $x^3 + 18x = 6$, to find the only real value of x.

$$\text{Here } \frac{2ar}{3b}\sqrt{\frac{a}{3}} = 2r\sqrt{6} = \tan. z,$$

Consequently, by formula 1, we shall have

Log. 6	0.7781513
$\frac{1}{2}$ log. 6	0.3890756
Log. 2	0.3010300
Log. r	10.0000000
Log. tan. z	10.6901056

Therefore $z = 78^\circ\ 27'\ 47''$, and $\frac{1}{2}z = 39^\circ\ 13'\ 53\frac{1}{2}''$.

Secondly, $\sqrt[3]{r^2 \tan. \frac{1}{2}z} = \tan. u$,

Whence

Log. r^2 . . .	20.0000000
Log. tan. $\frac{1}{2}z$ . .	9.9119523
Sum . . .	3)29.9119523
Log. tan. u . .	9.9706507

Therefore $u=43^\circ\ 3'\ 55''$, and $2u=86^\circ\ 7'\ 50''$.

Thirdly, $\frac{2}{r}\sqrt{\frac{a}{3}}\times\text{cot.}\,2u=\frac{2}{r}\sqrt{6}\times\text{cot.}\,2u$,

Whence

$\frac{1}{2}$ log. 6	0.3890756
Log. 2	0.3010300
Log. cot. $2u$ . .	8.8301834
Sum	9.5202890
Log. r	10.0000000
Log. x	$\bar{1}$.5202890

Consequently $x=.3313139$, the positive root of the equation, as was required.

2. Given $x^3-3x=1$, which is an equation falling under the irreducible case, to find its three roots.

Here, taking the radius $r=1$, we shall have,

$$\text{cos.}\,z=\frac{3b}{2a}\sqrt{\frac{3}{a}}=.5=\text{cos.}\,60^\circ,$$

Whence

$$x=2\sqrt{\frac{a}{3}}\times\text{cos.}\,\frac{z}{3}=2\,\text{cos.}\,20^\circ=+1.8793852,$$

$$x=-2\sqrt{\frac{a}{3}}\times\text{cos.}\,(60^\circ-\frac{z}{3})=-2\,\text{cos.}\,40^\circ=$$
$$-1.5320888,$$

$$x=-2\sqrt{\frac{a}{3}}\times\text{cos.}\,(60^\circ+\frac{z}{3})=-2\,\text{cos.}\,80^\circ=$$
$$-.3472964.$$

And in the equation $x^3-3x=-1$, the three roots, or values of x, are -1.8793852, 1.5320888, and $.3472964$, which are the negatives of the roots of the former equation.

3. Given $x^3-2x=-2$, to find the root of the equation, or the value of x.

Ans. $x=-1.7693$

4. Given $x^3-300x=-1000$, to find the root of the equation, or the value of x.

Ans. $x=3.472964$

5. Given $x^3-9x=9$, to find all the three roots, or values of x.

Ans. 3.411474, -2.226682, and -1.184792

6. Given $x^3-x^2-2x+1=0$, to find the three roots of the equation.

Ans. $2\cdot\cos.\frac{1}{7}\pi$, $-2\cos.\frac{2}{7}\pi$, and $2\cos.\frac{3}{7}\pi$,

Where $\pi=180^\circ$

Of the solution of cubic equations by converging series.

This method, which, in some cases, will be found more convenient in practice than either of the former, consists in substituting the numeral parts of the given equation, in the place of the literal, in one of the following general formulæ, according to which it may be found to belong, and then collecting as many terms of the series as are sufficient for determining the value of the unknown quantity, to the degree of exactness required.

1. $x^3+ax=b$.

$$x=\frac{2b}{\sqrt[3]{2(27b^2+4a^3)}}\Big\{1+\frac{2.5}{6.9}\Big(\frac{27b^2}{27b^2+4a^3}\Big)+\frac{2.5.8.11}{6.9.12.15}\Big(\frac{27b^2}{27b^2+4a^3}\Big)^2$$
$$+\frac{2.5.8.11.14.17}{6.9.12.15.18.21}\Big(\frac{27b^2}{27b^2+4a^3}\Big)^3\&c.\Big\}.$$

Or,

$$x=\frac{2b}{\sqrt[3]{2(27b^2+4a^3)}}\Big\{1+\frac{2.5}{6.9}\Big(\frac{27b^2}{27b^2+4a^3}\Big)\text{A}+\frac{8.11}{12.15}\Big(\frac{27b^2}{27b^2+4a^3}\Big)\text{B}$$
$$+\frac{14.17}{18.21}\Big(\frac{27b^2}{27b^2+4a^3}\Big)\text{C}\&c.\Big\}.$$

In which case, as well as in all the following ones, A, B, C, &c. denote the terms immediately preceding those in which they are first found.

$$2.\quad x^3-ax=\pm b$$

Where $\frac{1}{4}b^2$ is supposed to be greater than $\frac{1}{27}a^3$, or $27b^2 > 4a^3$.

$$x=\pm 2\sqrt[3]{\frac{b}{2}}\Big\{1-\frac{2}{3.6}\Big(\frac{27b^2-4a^3}{27b^2}\Big)-\frac{2.5.8}{3.6.9.12}\Big(\frac{27b^2-4a^3}{27b^2}\Big)^2-$$
$$\frac{2.5.8.11.14}{3.6.9.12.15.18}\Big(\frac{27b^2-4a^3}{27b^2}\Big)^3\&c.\Big\}.$$

Or,

$$x=\pm 2\sqrt[3]{\frac{b}{2}}\Big\{1-\frac{2}{3.6}\Big(\frac{27b^2-4a^3}{27b^2}\Big)\text{A}-\frac{5.8}{9.12}\Big(\frac{27b^2-4a^3}{27b^2}\Big)\text{B}-$$
$$\frac{11.14}{15.18}\Big(\frac{27b^2-4a^3}{27b^2}\Big)\text{C}\&c.\Big\}.$$

In which case the upper sign must be taken when b is positive, and the under sign when it is negative; and the same for the first roots in the two following cases.

$$3.\quad x^3-ax=\pm b$$

Where $\frac{1}{4}b^2$ is supposed to be less than $\frac{1}{27}a^3$, or $27b^2 < 4a^3$.

$$x=\pm 2\sqrt[3]{\frac{b}{2}}\Big\{1+\frac{2}{3.6}\Big(\frac{4a^3-27b^2}{27b^2}\Big)-\frac{2.5.8}{3.6.9.12}\Big(\frac{4a^3-27b^2}{27b^2}\Big)^2+$$
$$\frac{2.5.8.11.14}{3.6.9.12.15.18}\Big(\frac{4a^3-27b^2}{27b^2}\Big)^3-\&c.\Big\}.$$

Or,

$$x = \pm 2\sqrt[3]{\frac{b}{2}}\left\{1 + \frac{2}{3.6}\left(\frac{4a^3-27b^2}{27b^2}\right)\mathrm{A} - \frac{5.8}{9\cdot 12}\left(\frac{4a^3-27b^2}{27b^2}\right)\mathrm{B} + \frac{11.14}{15.18}\left(\frac{4a^3-27b^2}{27b^2}\right)\mathrm{C} - \&c.\right\}.$$

Which series answers to the irreducible case, and must be used when $2a^3$ is less than $27b^2$.

And if the root thus found be put $= r$, the other two roots may be expressed as follows:

$$x = \mp\frac{r}{2} \pm \frac{\sqrt{(4a^3-27b^2)}}{9\sqrt[3]{2b^2}}\left\{1 - \frac{2.5}{6.9}\left(\frac{4a^3-27b^2}{27b^2}\right) + \frac{2.5.8.11}{6.9.12.15}\left(\frac{4a^3-27b^2}{27b^2}\right)^2 - \frac{2.5.8.11.14.17}{3.6.9.12.15.18.21}\left(\frac{4a^3-27b^2}{27b^2}\right)^3 \&c.\right\}.$$

Or,

$$x = \mp\frac{r}{2} \pm \frac{\sqrt{(4a^3-27b^2)}}{9\sqrt[3]{2b^2}}\left\{1 - \frac{2.5}{6.9}\left(\frac{4a^3-27b^2}{27b^2}\right)\mathrm{A} + \frac{8.11}{12.15}\left(\frac{4a^3-27b^2}{27b^2}\right)\mathrm{B} - \frac{14.17}{18.21}\left(\frac{4a^3-27b^2}{27b^2}\right)\mathrm{C}\&c.\right\}.$$

Where $-\frac{1}{2}r$, or $+\frac{1}{2}r$, must be taken according as b is positive or negative; and the double sign ± 1 must be affixed to the other branch, as usual.

$$4.\quad x^3 - ax = \pm b$$

Where $\frac{1}{4}b^2$ is still supposed to be less than $\frac{1}{27}a^3$, or $27b^2 = 4a^3$.

$$x = \mp\frac{2b}{\sqrt[3]{2(4a^3-27b^2)}}\left\{1 - \frac{2.5}{6.9}\left(\frac{27b^2}{4a^3-27b^2}\right) + \frac{2.5.8.11}{6.9.12.5}\left(\frac{27b^2}{4a^3-27b^2}\right)^2 - \frac{2.5.8.11.14.17}{6.9.12.15.18.21}\left(\frac{27b^2}{4a^3-27b^2}\right)^3\&c.\right\}.$$

Or,

$$x = \mp\frac{2b}{\sqrt[3]{2(4a^3-27b^2)}}\left\{1 - \frac{2.5}{6.9}\left(\frac{27b^2}{4a^3-27b^2}\right)\mathrm{A} + \frac{8.11}{12.15}\left(\frac{27b^2}{4a^3-27b^2}\right)\mathrm{B} - \frac{14.17}{18.21}\left(\frac{27b^2}{4a^3-27b^2}\right)\mathrm{C}\&c.\right\}.$$

Which series also answers to the irreducible case, and must be used when $2a^3$ is greater than $27b^2$.

And if the root, thus found, be put $=r$, as before, the other two roots may be expressed thus:

$$x = \mp\frac{r}{2} \pm \sqrt[6]{\frac{4a^3-27b^2}{4}}\left\{1 + \frac{2}{3.6}\left(\frac{27b^2}{4a^3-27b^2}\right) - \frac{2.5.8}{3.6.9.12}\left(\frac{27b^2}{4a^3-27b^2}\right)^2 + \frac{2.5.8.11.14}{3.6.9.12.15.18.21}\left(\frac{27b^2}{4a^3-27b^2}\right)^3 \&c.\right\}.$$

Or,

$$x = \mp\frac{r}{2} \pm \sqrt[6]{\frac{4a^3-27b^2}{4}}\left\{1 + \frac{2}{3.6}\left(\frac{27b^2}{4a^3-27b^2}\right)\mathrm{A} - \frac{5.8}{9.12}\left(\frac{27b^2}{4a^3-27b^2}\right)\mathrm{B} + \frac{11.14}{18.21}\left(\frac{27b^2}{4a^3-27b^2}\right)\mathrm{C} \&c.\right\}.$$

Where the signs are to be taken as in the latter part of the preceding case.

EXAMPLES.

1. Given $x^3+6x=2$, to find the real value of x.

Here $a=6$, and $b=2$,

Whence

$$\frac{27b^2}{27b^2+4a^3} = \frac{27\times4}{27\times4+4\times216} = \frac{1}{1+8} = \frac{1}{9}; \text{ and } \frac{2b}{\sqrt[3]{2(27b^2+4a^3)}} =$$

$$\frac{4}{\sqrt[3]{2(4\times27+4\times216)}} = \frac{4}{2\sqrt[3]{27+8\times27}} = \frac{4}{6\sqrt[3]{9}} = \frac{2\sqrt[3]{81}}{27} = \frac{\sqrt[3]{648}}{27}.$$

And consequently we shall have, by formula 1,

1	1.0000000
$\frac{2.5}{6.9}\times\frac{1}{9}$	.0205761(A)
$\frac{8.11}{12.15}\times\frac{1}{9}\mathrm{A}$	.0011177(B)
$\frac{14.17}{18.21}\times\frac{1}{9}\mathrm{B}$	.0000782(C)
$\frac{20.23}{24.27}\times\frac{1}{9}\mathrm{C}$	.0000062(D)
$\frac{26.29}{30.33}\times\frac{1}{9}\mathrm{D}$	.0000005(E)
	1.0217787

Log. 1.0217787 . . .	0.0093570
Log. $\sqrt[3]{648}$ . . .	0.9371916
Colog. 27	8.5686362
N° .32748	$\bar{1}$.5151848

Therefore $x = .32748$.

2. Given $x^3 - 9x = 12$, to find the real value of x.

Here $a = 9$, and $b = 12$,

Whence $2\sqrt[3]{\frac{12}{2}} = 2\sqrt[3]{6}$, and $\frac{27b^2 - 4a^3}{27b^2} = \frac{27 \times 144 - 4 \times 27^2}{27 \times 144} = \frac{144 - 108}{144} = \frac{36}{144} = \frac{1}{4}$.

And consequently, by formula 2,

1	1.0000000
$-\frac{2}{3.6} \times \frac{1}{4}$	$-.0277778$(A)
$-\frac{5.8}{9.12} \times \frac{1}{4}A$	$-.0025720$(B)
$-\frac{11.14}{15.18} \times \frac{1}{4}B$	$-.0003667$(C)
$-\frac{17.20}{21.24} \times \frac{1}{4}C$	$-.0000619$(D)
$-\frac{23.26}{27.30} \times \frac{1}{4}D$	$-.0000114$(E)
$-\frac{29.32}{33.36} \times \frac{1}{4}E$	.0000022(F)
Sum	$-.0307920$
Compt	.9692080

Log. 969208 . . .	$\bar{1}$.9864137
Log. $2\sqrt[3]{6}$, or log. $\sqrt[3]{48}$	0.5604137
N° 3.522334	.5468274

Therefore $x = 3.522334$.

3. Given $x^3-12x=15$, to find the three values of x.

Here $a=12$, and $b=15$,

Whence $2\sqrt[3]{\frac{b}{2}}=2\sqrt[3]{\frac{15}{2}}=\sqrt[3]{60}$, and $\frac{4a^3-27b^2}{27b^2}=$ $\frac{4.12^3-27.15^2}{27.15^2}=\frac{256-225}{225}=\frac{31}{225}$

And consequently, by formula 3,

1	$+1.0000000$
$+\frac{2}{3.6}\times\frac{31}{225}$	$+0.0153086$(A)
$-\frac{5.8}{9.12}\times\frac{31}{225}$A	-0.0007812(B)
$+\frac{11.14}{15.18}\times\frac{31}{225}$B	$+0.0000614$(C)
$-\frac{17.20}{21.24}\times\frac{31}{225}$C . . .	-0.0000057(D)
$+\frac{23.26}{27.30}\times\frac{31}{225}$D	$+0.0000006$(E)
Sum of + terms . .	$+1.0153706$
Sum of − terms . .	$-.0007869$
Difference	1.0145837

Log. 1.0145837	.0062880
Log. $\sqrt[3]{60}$	.5927171
N° 3.971963	.5990051

Therefore the 1st (or affirmative) value of x is 3.971963(r).

Again, $\frac{\sqrt{4a^3-27b^2}}{9\sqrt[3]{2b^2}}=\frac{\sqrt{837}}{9\sqrt[3]{450}}=\frac{\sqrt{9\times93}}{9\sqrt[3]{450}}=\frac{\sqrt{93}}{3\sqrt[3]{450}}$; and $\frac{4a^3-27b^2}{27b^2}=\frac{31}{225}$.

Hence 1	1.0000000
$-\frac{2.5}{6.9}\times\frac{31}{225}$ . . .	$-.0267480$(A)
$+\frac{8.11}{12.15}\times\frac{31}{225}A$ . .	$+.0018017$(B)
$-\frac{14.17}{18.21}\times\frac{31}{225}B$ . .	$-.0001563$(C)
$+\frac{20.23}{24.27}\times\frac{31}{225}C$ . .	$+.0000153$(D)
$-\frac{26.29}{30.33}\times\frac{31}{225}D$ . .	$-.0000016$(E)
Sum	.9749111

Log. .9749111 . . .	$\bar{1}.9889694$
Log. $\sqrt{93}$	0.9842414
Colog. $\sqrt[3]{450}$ . .	9.1155959
Colog. 3	9.5228787
N° .408949	$\bar{1}.6116854$

Therefore $-\frac{p}{2}$ . .	-1.985981
Last number . .	± 0.408949
Result	-1.577032
Or	-2.394930

Whence the two negative values of x are -1.577032 and -2.394930.

4. Given $x^3-6x=2$, to find the three values of x.

Here $\frac{-2b}{\sqrt[3]{2(4a^3-27b^2)}} = \frac{-4}{\sqrt[3]{2(4.6^3-27.4)}} = \frac{-4}{3\sqrt[3]{2(4.8-4)}} =$ $\frac{-4}{6\sqrt[3]{7}} = \frac{-2}{3\sqrt[3]{7}} = -\frac{2\sqrt[3]{49}}{21}$; & $\frac{27b^2}{4a^3-27b^2} = \frac{4.27}{4.6^3-27.4} = \frac{1}{8-1} = \frac{1}{7}$

Hence, by formula 4,

1	1.0000000
$-\frac{2.5}{6.9}\times\frac{1}{7}$	$-.0264550$(A)
$+\frac{8.11}{12.15}\times\frac{1}{7}\text{A}$	$+.0018476$(B)
$-\frac{14.17}{18.21}\times\frac{1}{7}\text{B}$	$-.0001662$(C)
$+\frac{20.23}{24.27}\times\frac{1}{7}\text{C}$	$+.0000168$(D)
$-\frac{26.29}{30.33}\times\frac{1}{7}\text{D}$	$-.0000018$(E)
Sum	$+.9752414$

Log. .9752414	$\bar{1}.9891120$
Log. 2	0.3010300
Log. $\sqrt[3]{49}$	0.5633987
Colog. 21.	8.6777807
N°. 339870	$\bar{1}.5313214$

Consequently one of the negative roots is $-.339870\,(r)$.

Again, $\sqrt[6]{\frac{4a^3-27b^2}{4}}=\sqrt[6]{\frac{4.6^3-27.4}{4}}=\sqrt[6]{(6^3-27)}=\sqrt[6]{189}$;

and $\frac{27b^2}{4a^3-27b^2}=\frac{1}{7}$

Hence 1	1
$+\frac{2}{3.6}\times\frac{1}{7}$	$+.0158730$(A)
$-\frac{5.8}{9.12}\times\frac{1}{7}\text{A}$	$-.0008398$(B)
$+\frac{11.14}{15.18}\times\frac{1}{7}\text{B}$	$+.0000684$(C)
$-\frac{17.20}{21.24}\times\frac{1}{7}\text{C}$	$-.0000066$(D)
$+\frac{23.26}{27.30}\times\frac{1}{7}\text{D}$	$+.0000002$(E)
Sum	1.0150952

Log. 1.0150952 . . .	.0065070
Log. $\sqrt[6]{189}$	.3794103
N° 2.431741 . . .	.3859173
Therefore $\frac{r}{2}$	$+.169935$
Last number . . .	± 2.431741
Result	$+2.601676$
Or	-2.261806

And consequently $+2.601676$, -2.261806, and $-.339870$ are the three roots required.

5. Given $x^3+9x=30$, to find the root of the equation, or the value of x.

Ans. $x=2.180849$

6. Given $x^3-2x=5$, to find the root of the equation, or the value of x.

Ans. $x=2.0945514$

7. Given $x^3-23x=-16$, to find the root of the equation, or the value of x,

Ans. $x=-5.472136$

8. Given $x^3-27x=36$, to find the three roots or values of x.

Ans. 5.765722, -4.320684, and -1.445038

OF BIQUADRATIC EQUATIONS.

(R) *A biquadratic equation*, as before observed, is one that rises to the fourth power, or that is of the general form

$$x^4+ax^3+bx^2+cx+d=0,$$

the root of which may be determined by means of the following formulæ; substituting the numbers of the given equation, with their proper signs, in the places of the literal coefficients *a*, *b*, *c*, *d*.

RULE I.

Find the value of z in the cubic equation

$$z^3 + (\tfrac{1}{4}ac - \tfrac{1}{12}b^2 - d)z = \tfrac{1}{108}b^3 + \tfrac{1}{8}(c^2 + da^2) - \tfrac{b}{24}(ac + 8d)$$

by one of the former rules; and let the root, thus determined, be denoted by r.

Then find the two values of x in each of the following quadratic equations.

$$x^2 + (\tfrac{1}{2}a + \surd\{\tfrac{1}{4}a^2 + 2(r - \tfrac{1}{3}b)\})x = -(r + \tfrac{1}{6}b) + \surd\{(r + \tfrac{1}{6}b)^2 - d\}$$

$$x^2 + (\tfrac{1}{2}a - \surd\{\tfrac{1}{4}a^2 + 2(r - \tfrac{1}{3}b)\})x = -(r + \tfrac{1}{6}b) - \surd\{(r + \tfrac{1}{6}b)^2 - d\}$$

and they will be the four roots of the biquadratic equation required.

Or, if the equation be of the more commodious form

$$x^4 + bx^2 + cx + d = 0,$$

to which it can be always reduced, by taking away its second term, the rule will be as follows:

Find the value of z in the cubic equation

$$z^3 - (\tfrac{1}{12}b^2 + d)z = \tfrac{1}{108}b^3 + \tfrac{1}{8}c^2 - \tfrac{1}{3}bd,$$

and let the root thus determined be denoted by r.

Then find the two values of x, in each of the following quadratic equations,

$$x^2 + (\surd\{2(r - \tfrac{1}{3}b)\})x = -(r + \tfrac{1}{6}b) + \surd\{(r + \tfrac{1}{6}b)^2 - d\}$$

$$x^2 - (\surd\{2(r - \tfrac{1}{3}b)\})x = -(r + \tfrac{1}{6}b) - \surd\{(r + \tfrac{1}{6}b)^2 - d\}$$

and they will be the four roots of the biquadratic equation required.

Or the four roots of the given equation, in this last case, will be as follows:

$$x = -\tfrac{1}{2}\sqrt{\{2(r-\tfrac{1}{3}b)\}} + \sqrt{\{-\tfrac{r}{2}-\tfrac{b}{3}+\sqrt{[(r+\tfrac{1}{6}b)^2-d]}\}}$$

$$x = -\tfrac{1}{2}\sqrt{\{2(r-\tfrac{1}{3}b)\}} - \sqrt{\{-\tfrac{r}{2}-\tfrac{b}{3}+\sqrt{[(r+\tfrac{1}{6}b)^2-d]}\}}$$

$$x = +\tfrac{1}{2}\sqrt{\{2(r-\tfrac{1}{3}b)\}} + \sqrt{\{-\tfrac{r}{2}-\tfrac{b}{3}-\sqrt{[(r+\tfrac{1}{6}b)^2-d]}\}}$$

$$x = +\tfrac{1}{2}\sqrt{\{2(r-\tfrac{1}{3}b)\}} - \sqrt{\{-\tfrac{r}{2}-\tfrac{b}{3}-\sqrt{[(r+\tfrac{1}{6}b)^2-d]}\}}$$

(b)

EXAMPLES.

1. Given the equation $x^4-10x^3+35x^2-50x+24=0$, to find its roots.

Here $a=-10$, $b=35$, $c=-50$, and $d=24$; Whence, by substituting these numbers in the cubic equation

$$z^3+(\tfrac{1}{4}ac-\tfrac{1}{12}b^2-d)z=\tfrac{1}{108}b^3+\tfrac{1}{8}(c^2+da^2)-\tfrac{b}{24}(ac+8d),$$

we shall have

$$z^3-\frac{13}{12}z=\frac{35}{108},$$

which equation, being solved according to the rule before given for that purpose, gives

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(b) Biquadratic equations, in certain particular cases, are reducible to quadratics, as may be readily shown from the demonstration of the general rule, given in Vol. II of the present work; but as such equations seldom occur in practice, and are of little importance as an object of investigation, any formal enumeration of the methods that might be given for resolving them would be unnecessary; particularly as it can be easily discovered, from the relation of the coefficients, whether any given numeral equation is of this kind or not.

$$z=\frac{1}{6}\{\sqrt[3]{(35+18\sqrt{-3})}+\sqrt[3]{(35-18\sqrt{-3})}\}$$

But, by the rule for binomial surds, given in the Note to Art. I. Case II,

$$\sqrt[3]{(35+18\sqrt{-3})}=\frac{7}{2}+\frac{1}{2}\sqrt{-3},$$

$$\text{And } \sqrt[3]{(35-18\sqrt{-3}}=\frac{7}{2}-\frac{1}{2}\sqrt{-3};$$

$$\text{Therefore } z=\frac{1}{6}\{\frac{7}{2}+\frac{1}{2}\sqrt{-3}+\frac{7}{2}-\frac{1}{2}\sqrt{-3}\}=\frac{7}{6}$$

And if this number be substituted for r, -10 for a, 35 for b, and 24 for d, in the two quadratic equations

$$x^2+[\frac{1}{2}a+\sqrt{\frac{1}{4}}\{a^2+2(r-\frac{1}{3}b)\}]x=-(r+\frac{1}{6}b)+\sqrt{\{(r+\frac{1}{6}b)^2-d\}}$$

$$x^2+[\frac{1}{2}a-\sqrt{\frac{1}{4}}\{a^2+2(r-\frac{1}{3}b)\}]x=-(r+\frac{1}{6}b)-\sqrt{\{(r+\frac{1}{6}b)^2-d\}},$$

they will become, after proper reduction,

$$x^2-3x=-2, \text{ and } x^2-7x=-12:$$

$$\text{From the first of which } x=\frac{3}{2}\pm\sqrt{\frac{1}{4}}=\frac{3}{2}\pm\frac{1}{2}=2 \text{ or } 1,$$

$$\text{And from the second } x=\frac{7}{2}\pm\sqrt{\frac{1}{4}}=\frac{7}{2}\pm\frac{1}{2}=4 \text{ or } 3;$$

Whence the four roots of the given equation are 1, 2, 3, and 4.

2. Given $x^4+12x-17=0$, to find the four roots of the equation.

Here $a=0$, $b=0$, $c=12$, and $d=-17$;

Whence, by substituting these numbers in the cubic equation

$$z^3 - (\tfrac{1}{12}b^2 + d)z = \tfrac{1}{108}b^3 + \tfrac{1}{8}c^2 - \tfrac{1}{3}bd,$$

we shall have

$$z^3 + 17z = 18,$$

Where it is evident, by inspection, that $z = 1$.

And if this number be substituted for r, 0 for b, and -17 for d in the four expressions for x, given above, they will become

$$x = -\tfrac{1}{2}\sqrt{2} + \sqrt{-\tfrac{1}{2} + \sqrt{18}} = -\tfrac{1}{2}\sqrt{2} + \sqrt{-\tfrac{1}{2} + 3\sqrt{2}}$$
$$x = -\tfrac{1}{2}\sqrt{2} - \sqrt{-\tfrac{1}{2} + \sqrt{18}} = -\tfrac{1}{2}\sqrt{2} - \sqrt{-\tfrac{1}{2} + 3\sqrt{2}}$$
$$x = +\tfrac{1}{2}\sqrt{2} + \sqrt{-\tfrac{1}{2} - \sqrt{18}} = +\tfrac{1}{2}\sqrt{2} + \sqrt{-\tfrac{1}{2} - 3\sqrt{2}}$$
$$x = +\tfrac{1}{2}\sqrt{2} - \sqrt{-\tfrac{1}{2} - \sqrt{18}} = +\tfrac{1}{2}\sqrt{2} - \sqrt{-\tfrac{1}{2} - 3\sqrt{2}}$$

Which are the four roots of the proposed equation; the two first being real, and the two latter imaginary.

RULE II.

The roots of any biquadratic equation of the form

$$x^4 + ax^2 + bx + c = 0,$$

may also be determined by the following general formulæ first given by Euler; which are remarkable for their elegance and simplicity.

Find the three roots of the cubic equation

$$z^3 + 2az^2 + (a^2 - 4c)z = b^2$$

by either of the former rules, before given, for this purpose; and let them be denoted by r', r'', and r'''.

Then, we shall have

When b is positive

$$x = \tfrac{1}{2}\{-\sqrt{r'} - \sqrt{r''} - \sqrt{r'''}\}$$
$$x = \tfrac{1}{2}\{-\sqrt{r'} + \sqrt{r''} + \sqrt{r'''}\}$$
$$x = \tfrac{1}{2}\{+\sqrt{r'} - \sqrt{r''} + \sqrt{r'''}\}$$
$$x = \tfrac{1}{2}\{+\sqrt{r'} + \sqrt{r''} - \sqrt{r'''}\}$$

When b is negative.

$$x=\tfrac{1}{2}\{+\sqrt{r'}+\sqrt{r''}+\sqrt{r'''}\}$$
$$x=\tfrac{1}{2}\{+\sqrt{r'}-\sqrt{r''}-\sqrt{r'''}\}$$
$$x=\tfrac{1}{2}\{-\sqrt{r'}+\sqrt{r''}-\sqrt{r'''}\}$$
$$x=\tfrac{1}{2}\{-\sqrt{r'}-\sqrt{r''}+\sqrt{r'''}\}$$

Where it is to be observed, that if the three roots of the cubic equation, r', r'', r''', be all real and positive, the four roots of the proposed equation will, also, be real; and if one of these roots be positive, and the other two imaginary, or both of them negative, and equal to each other, two of the roots of the given equation will be real, and two imaginary; which are the only cases that produce real results.

3. Given $x^4-25x^2+60x-36=0$, to find the four roots of the equation.

Here $a=-25$, $b=60$, and $c=-36$;

Whence, by substituting these values for their equals, in the cubic equation above given, we shall have

$$z^3-2\times 25z^2+(25^2+4\times 36)z=60^2, \text{ or}$$
$$z^3-50z^2+769z=3600:$$

The three roots of which last equation, as found by trial, or by either of the former rules, are 9, 16, and 25, respectively; whence

$$x=\tfrac{1}{2}(-\sqrt{9}-\sqrt{16}-\sqrt{25})=\tfrac{1}{2}(-3-4-5)=-6$$
$$x=\tfrac{1}{2}(-\sqrt{9}+\sqrt{16}+\sqrt{25})=\tfrac{1}{2}(-3+4+5)=+3$$
$$x=\tfrac{1}{2}(+\sqrt{9}-\sqrt{16}+\sqrt{25})=\tfrac{1}{2}(+3-4+5)=+2$$
$$x=\tfrac{1}{2}(+\sqrt{9}+\sqrt{16}-\sqrt{25})=\tfrac{1}{2}(+3+4-5)=+1$$

And consequently the four roots of the proposed equation are 1, 2, 3, and -6.

EXAMPLES FOR PRACTICE.

1. Given $x^4-55x^2-30x+504=0$, to find the four roots, or values of x.

Ans. 3, 7, −4, and −6

2. Given $x^4+2x^3-7x^2-8x=-12$, to find the four roots, or values of x.

Ans. 1, 2, −3, and −2

3. Given $x^4-8x^3+14x^2+4x=8$, to find the four roots, or values of x.

Ans. $\left\{\begin{array}{ll} 3+\sqrt{5}, & 3-\sqrt{5} \\ 1+\sqrt{3}, & 1-\sqrt{3} \end{array}\right.$

4. Given $x^4-17x^2-20x-6=0$, to find the four roots, or values of x.

Ans. $\left\{\begin{array}{ll} 2+\sqrt{7}, & 2-\sqrt{7} \\ -2+\sqrt{2}, & -2-\sqrt{2} \end{array}\right.$

5. Given $x^4-3x^2-4x=3$, to find the four roots, or values of x.

Ans. $\left\{\begin{array}{ll} \frac{1}{2}+\frac{1}{2}\sqrt{3}, & \frac{1}{2}-\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}+\frac{1}{2}\sqrt{3}, & -\frac{1}{2}-\frac{1}{2}\sqrt{3} \end{array}\right.$

6. Given $x^4-19x^3+132x^2-302x+200=0$, to find the four roots of the equation.

Ans. $\left\{\begin{array}{ll} 1.02804, & 4.00000 \\ 6.57653, & 7.39543 \end{array}\right.$

7. Given $x^4-27x^3+162x^2+356x-1200=0$, to find the four roots of the equation.

Ans. $\left\{\begin{array}{ll} 2.05608, & -3.00000 \\ 13.15306, & 14.79086 \end{array}\right.$

8. Given $x^4-12x^2+12x-3=0$, to find the four roots of the equation.

Ans. $\left\{\begin{array}{ll} .606018, & -3.907378 \\ 2.858083, & .443277 \end{array}\right.$

9. Given $x^4-36x^2+72x-36=0$, to find the four roots of the equation.

Ans. $\begin{cases} 0.8729836, & 1.2679494 \\ 4.7320506, & -6.8729836 \end{cases}$

10 Given $x^4-5x^3+13x^2-17x+12=0$, to find the four roots of the equation, which are all imaginary.

OF THE RESOLUTION OF EQUATIONS BY APPROXIMATION, OR THE METHOD OF SUCCESSIVE SUBSTITUTIONS.

(s) Equations of the fifth power, and those of higher dimensions, cannot be resolved by any rule or algebraical formula, that has yet been discovered; except in some particular cases, where certain relations subsist between the coefficients of their several terms, or when the roots are rational, and, for that reason, can be easily found by means of a few trials. In these cases, therefore, recourse must be had to the following method of approximation, which is universally applicable to all kinds of numeral equations, whatever may be the number of their dimensions, and, though not strictly accurate, will give the value of the root sought, to any required degree of exactness (c).

(c) The first attempt to determine the roots of equations, by approximation, appears to have been made by Vieta, in his treatise *de Numeros a potestatum adfectarum Resolutione,* where he follows a process similar to that now used for obtaining the roots of numbers; which method was afterwards improved by Harriot, Oughtred, Pell, and others, who rendered it something more simple; but the multitude of operations that were required to be performed, and the uncertainty of success in a great number of cases, occasioned it to be entirely abandoned before the end of the seven-

RULE.

Find, by trials, a number nearly equal to the root sought, which call r; and let z be made to denote the difference between this assumed root, and the true root x.

Then, instead of x, in the given equation, substitute its equal $r \pm z$, and there will arise a new equation, involving only z and known quantities.

Reject all the terms of this equation in which z is of two or more dimensions; and the approximate value of z may then be determined by means of a simple equation.

And if the value, thus found, be added to, or subtracted from that of r, according as r was assumed too little, or too great, it will give a near value of the root required.

But as this approximation will seldom be sufficiently exact, the operation must be repeated, by substituting the number, thus found, for r, in the equation exhibiting the value of z; when a second correction of z will be obtained, which, being added

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teenth century. Since that time Simpson (in his Select Exercises) and several later writers, have employed different methods for the same purpose; but the rule given in the text, which is that of Newton, as improved by Ralphson, will be found as commodious, and easy in its application, as any that has yet been proposed.

Its chief defect, which likewise attends the other methods here mentioned, is, that it does not show the progress made in the approximation at each operation; and that, when some of the roots are nearly equal to each other, they may be passed over without being perceived; both of which circumstances have been particularly noticed by Lagrange; who has given a new and improved method of approximation, in his late work *de la Résolution des E'quations Numériques*, to which we must refer the reader, to, or subtracted from r, will give a nearer value of the root than the former.

And by again substituting this last number for r, in the above mentioned equation, and repeating the same process as often as may be thought necessary, a value of x may be found to any degree of accuracy required.

Note. The decimal part of the root, as found both by this and the next rule, will, in general, about double itself at each operation; and therefore it would be useless, as well as troublesome, to use a much greater number of figures than these in the several substitutions for the values of r.

EXAMPLES.

1. Given $x^3+x^2+x=90$, to find the value of x, by approximation.

Here the root, as found by a few trials, is nearly equal to 4.

Let therefore $4=r$, and $r+z=x$;

$$\text{Then}\left|\begin{array}{l} x^3=r^3+3r^2z+3rz^2+z^3 \\ x^2=r^2+2rz+z^2 \\ x=r+z \end{array}\right|=90.$$

And by rejecting the terms $z^3+3rz^2+z^2$, as small in comparison with z, we shall have

$$r^3+r^2+r+3r^2z+2rz+z=90;$$

$$\text{Hence } z=\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-64-16-4}{48+8+1}=\frac{6}{57}=.10$$

And consequently $x=4.1$, *nearly*.

Again, if 4.1 be substituted in the place of r, in the last equation, we shall have

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as being but little susceptible of that kind of concise elucidation, which belongs to a performance like the present.

$$\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-68.921-16.81-4.1}{50.43+8.2+1}=.00283$$

And consequently $x=4.1+.00283=4.10283$, *nearly*.

And, if this number be again substituted for r, in the same equation, a still nearer value of the root will be obtained; and so on, as far as may be thought necessary.

2. Given $x^2+20x=100$, to find the value of x by approximation. Ans. $x=4.1421356$

3. Given $x^3+9x^2+4x=80$, to find the value of x by approximation. Ans. $x=2.4721359$

4. Given $x^4-38x^3+210x^2+538x+289=0$, to find the value of x by approximation.

Ans. $x=30.53565375$

5. Given $x^5+6x^4-10x^3-112x^2-207x-110=0$, to find the value of x by approximation.

Ans. 4.46410161

The roots of equations, of all orders, can also be determined, to any degree of exactness, by means of the following easy rule of double position; which, though it has not been commonly employed for this purpose, will be found, in some respects, superior to the former, as it can be applied, at once, to any unreduced equation, consisting of surds, or compound quantities, as readily as if it had been brought to its usual form.

RULE.

Find, by trial, two numbers as near the true root as possible, and substitute them in the given equation instead of the unknown quantity, noting the results that are obtained from each.

Then, as the difference of these results is to the difference of the two assumed numbers, so is the difference between the true result, and either of the former, to the correction of the number belonging to the result used; which correction being added to that number when it is too little, or subtracted from it when it is too great, will give the root required, *nearly*.

And if the number thus determined, and the nearest of the two former, or any other that appears to be more accurate, be now taken as the assumed roots, and the operation repeated as before, a new value of the unknown quantity will be obtained still more correct than the first; and so on, proceeding in this manner, as far as may be judged necessary (*d*).

EXAMPLES.

1. Given $x^3+x^2+x=100$, to find an approximate value of x.

(*d*) The above rule for Double Position, which is by far the most simple and commodious of any that has yet been devised for this purpose, is the same as that which was first given at p. 311 of the octavo edition of my Arithmetic, published in 1810.

To this we may farther add, that when one of the roots of an equation has been found, either by this method or the former, the rest may be determined as follows:

Bring all the terms to the left hand side of the equation, and divide the whole expression, so formed, by the difference between the unknown quantity (x) and the root first found; and the resulting equation will then be depressed a degree lower than the given one.

Find a root of this new equation, by approximation, as in

Here it is soon found, by a few trials, that the value of x lies between 4 and 5.

Hence, by taking these as the two assumed numbers, the operation will be as follows:

First Sup.		*Second Sup.*
4	x	5
16	x^2	25
64	x^3	125
84	Results	155

Therefore	155	5	100		
	84	4	84		
	71 :	1 ::	16 :	.225	

And consequently $x = 4 + .225 = 4.225$, *nearly*.

Again, if 4.2 and 4.3 be taken as the two assumed numbers, the operation will be as follows:

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first instance, and the number so obtained will be a second root of the original equation.

Then, by means of this root, and the unknown quantity, depress the second equation a degree lower, and thence find a third root; and so on, till the equation is reduced to a quadratic; when the two roots of this, together with the former, will be the roots of the equation required.

Thus, in the equation $x^3 - 15x^2 + 63x = 50$, the first root is found, by approximation, to be 1.02804. Hence

$x - 1.02804)x^3 - 15x^2 + 63x - 50(x^2 - 13.97196x + 48.63627 = 0.$

And the two roots of the quadratic equation, $x^2 - 13.97196x = -48.63627$, found in the usual way, are 6.57653 and 7.39543.

So that the three roots of the given cubic equation $x^3 - 15x^2 + 63x = 50$, are 1.02804, 6.57653, and 7.39543; their sum being $= 15$, the coefficient of the second term of the equation, as it ought to be when they are right.

First Sup.		*Second Sup.*
4.2	. . x . .	4.3
17.64	. . x^2 . .	18.49
74.088	. . x^3 . .	79.507
95.928	Results	102.297

Therefore

102.297	. .	4.3	. .	102.297		
95.928	. .	4.2	. .	100		
6.369	:	.1	::	2.297	:	.036

And consequently $x = 4.3 - .036 = 4.264$, *nearly*.

Again, let 4.264 and 4.265 be the two assumed numbers; then

First Sup.		*Second Sup.*
4.264	. . x . .	4.265
18.181696	. . x^2 . .	18.190225
77.526752	. . x^3 . .	77.581310
99.972448	Results	100.036535

Therefore

100.036535		4.265		100		
99.972448		4.264		99.972448		
.064087	:	.001	::	.027552	:	.0004299

And consequently

$$x = 4.264 + .0004299 = 4.2644299, \textit{ very nearly.}$$

2. Given $(\frac{1}{5}x^2 - 15)^2 + x\sqrt{x} = 90$, to find an approximate value of x.

Here, by a few trials, it will be soon found, that the value of x lies between 10 and 11; which let, therefore, be the two assumed numbers, agreeably to the directions given in the rule.

Then

First Sup.		Second Sup.
25	$(\frac{1}{5}x^2-15)^2$	84.64
31.622	$x\sqrt{x}$	36.482
56.622	Results	121.122

Hence

121.122	11	121.122
56.622	10	90
64.5 :	1 ::	31.122 : .482

And consequently $x=11-.482=10.518$

Again, let 10.5 and 10.6 be the two assumed numbers,

Then

First Sup.		Second Sup.
49.7025	$(\frac{1}{5}x^2-15)^2$	55.830784
34.0239	$x\sqrt{x}$	34.511099
83.7264	Results	90.341883

Hence

90.341883	10.6	90.341883
83.7264	10.5	90.
6.615483 :	.1 ::	.341883 : .0051679

And consequently

$x=10.6-.0051679=10.5948321$, *very nearly.*

3. Given $x^3+10x^2+5x=2600$, to find a near approximate value of x. Ans. $x=11.00673$

4. Given $2x^4-16x^3+40x^2-30x+1=0$, to find a near value of x. Ans. $x=1.284724$

5. Given $x^5+2x^4+3x^3+4x^2+5x=54321$, to find the value of x. Ans. 8.414455

6. Given $\sqrt[3]{7x^3+4x^2}+\sqrt{20x^2-10x}=28$, to find the value of x. Ans. 4.510661

7. Given $\sqrt{144x^2-(x^2+20)^2}+\sqrt{196x^2-(x^2+24)^2}=114$, to find the value of x. Ans. 7,123883

OF RECIPROCAL EQUATIONS.

(T) ALTHOUGH no general method has hitherto been discovered, for solving equations higher than those of the 4th power, there are, notwithstanding, some particular equations, of all orders under the 10th, which, on account of the relations subsisting between their coefficients, may be resolved by the rules that have been already given for the first four orders.

This is particularly the case with what have been usually called reciprocal equations; which are such that the coefficients of their several terms form the same numerical series, whether taken forwards or backwards; or that remain of the same form when the reciprocal of the unknown quantity, or $\frac{1}{x}$, is substituted for x; except that the terms are then reversed. Thus the equations

$$x^{2m}+px^{2m-1}+qx^{2m-2}+ \ldots +qx^2+px+1=0,$$

And

$$x^{2m+1}+px^{2m}+qx^{2m-1}+ \ldots +qx^2+px+1=0,$$

may always be transformed into others of a degree which is denoted by half the exponent of the highest power of the unknown quantity, if it be an even number, or by half that exponent when diminished by 1, if it be an odd number; the method of resolving them, as far as to the 9th order, inclusively, being as follows (*e*).

(*e*) From the property of equations of this kind not being

RULE.

When the index of the highest term is an even number.

1. If the equation be of the 4th power, as $x^4+px^3+qx^2+px+1=0$, find the two values of z in the equation $z^2+pz+q-2=0$, and let them be denoted by r and r'; then the roots of the two quadratic equations $x^2-rx+1=0$, and $x^2-r'x+1=0$, will be the four roots of the equation proposed.

2. If it be of the 6th power, as
$$x^6+px^5+qx^4+rx^3+qx^2+px+1=0,$$
find the three values of z in the cubic equation $z^3+pz^2+(q-3)z+r-2p=0$, and let them be denoted by r, r', r''; then the roots of the three quadratic equations $x^2-rx+1=0$, $x^2-r'x+1=0$, and $x^2-r''x+1=0$, will be the six roots of the proposed equation.

3. And if it be of the 8th power, as
$$x^8+px^7+qx^6+rx^5+sx^4+rx^3+qx^2+px+1=0,$$
find the four values of z in the biquadratic equation $z^4+pz^3+(q-4)z^2+(r-3p)z+s-2(q-1)=0$, and let them be denoted by r, r', r'', r'''; then the

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changed by substituting $\frac{1}{x}$ for x, it follows, that if any quantity a be one of the roots, $\frac{1}{a}$ will also be a root; and consequently as -1 is always a root of the equation when the number of its dimensions is odd, it is plain that every reciprocal equation of the $2m$th, or $2m+1$, order, can always be reduced to another of the mth order; for a demonstration of which, and other particulars, see the Miscellanea Analytica of De Moivre, page 71, and Waring's Medit. Algebraicæ, c. 3; also the corresponding article (r) in Vol. II of the present performance.

roots of the four quadratic equations, $x^2 - rx + 1 = 0$, $x^2 - r'x + 1 = 0$, $x^2 - r''x + 1 = 0$, and $x^2 - r'''x + 1 = 0$, will be the eight roots of the proposed equation.

When the index of the highest term is an odd number.

1. If the equation be of the 3d power, as

$$x^3 + px^2 + px + 1 = 0,$$

where one of its roots is evidently -1, find the value of z in the simple equation $z + p - 1 = 0$, and let it be denoted by r; then the roots of the quadratic equation $x^2 - rx + 1 = 0$, will be the other two roots of the proposed equation.

2. If it be of the 5th power, as

$$x^5 + px^4 + qx^3 + qx^2 + px + 1 = 0,$$

where one of its roots is also -1, find the two values of z in the equation $z^2 + (p - 1)z + q - p - 1 = 0$, and let them be denoted by r and r'; then the roots of the two quadratic equations $x^2 - rx + 1 = 0$, and $x^2 - r'x + 1 = 0$, will be the other four roots of the proposed equation.

3. If it be of the 7th power, as

$$x^7 + px^6 + qx^5 + rx^4 + rx^3 + qx^2 + px + 1 = 0,$$

where one of the roots is -1, as before, find the three values of z in the cubic equation

$$z^3 + (p - 1)z^2 + (q - p - 2)z + r - p - q + 1 = 0,$$

and let them be denoted by r, r', r''; then the roots of the three quadratic equations $x^2 - rx + 1 = 0$, $x^2 - r'x + 1 = 0$, and $x^2 - r''x + 1 = 0$, will be the other six roots of the proposed equation.

3. And if it be of the 9th power, as

$$x^9 + px^8 + qx^7 + rx^6 + sx^5 + sx^4 + rx^3 + qx^2 + px + 1 = 0,$$

where one of the roots is -1, as it is in all the odd powers, find the four values of z in the biquadratic equation

$$z^4 + (p-1)z^3 + (q-p-3)z^2 + (r-q-2p+2)z + s-r-q+p+1=0,$$

and let them be denoted by r, r', r'', r'''; then the roots of the four quadratic equations $x^2 - rx + 1 = 0$, $x^2 - r'x + 1 = 0$, $x^2 - r''x + 1 = 0$, and $x^2 - r'''x + 1 = 0$, will be the other eight roots of the proposed equation.

Where it is to be observed, that, in all the cases here mentioned, the coefficients p, q, r, s, of the original equation, may be either positive or negative; but the first and last terms must be positive.

EXAMPLES.

1. Given $x^4 + 4x^3 - 19x^2 + 4x + 1 = 0$, to find the four roots of the equation.

Here $p = 4$ and $q = -19$,

Hence $z^2 + pz = 2 - q$ becomes $z^2 + 4z = 2 + 19 = 21$,

Where the two values of z are $+3$ and -7;

Consequently $x^2 - 3x = -1$, and $x^2 + 7x = -1$,

From the first of which equations $x = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}$,

And from the second $x = -\frac{7}{2} \pm \frac{3}{2}\sqrt{5}$;

Therefore the four roots of the proposed equation are,

$$\frac{3}{2} + \frac{1}{2}\sqrt{5},\ \frac{3}{2} - \frac{1}{2}\sqrt{5},\ -\frac{7}{2} + \frac{3}{2}\sqrt{5},\ \text{and}\ -\frac{7}{2} - \frac{3}{2}\sqrt{5}.$$

2. Given $x^5 - 11x^4 + 17x^3 + 17x^2 - 11x + 1 = 0$, to find the five roots of the equation.

Here $p = -11$, and $q = 17$,

Hence

$z^2 + (p-1)z = 1 + p - q$ becomes $z^2 - 12x = -27$,

Where the two values of z are 9 and 3,
Consequently $x^2-9x=-1$, and $x^2-3x=-1$;
From the first of which equations $x=\frac{9}{2}\pm\frac{1}{2}\sqrt{77}$,

And from the second $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$;

Therefore the five roots of the proposed equation are

$$-1, \frac{9}{2}+\frac{1}{2}\sqrt{77}, \frac{9}{2}-\frac{1}{2}\sqrt{77}, \frac{3}{2}+\frac{1}{2}\sqrt{5}, \text{ and } \frac{3}{2}-\frac{1}{2}\sqrt{5}.$$

Note. If the coefficients of either of the orders of equations, mentioned in the rule, be in part literal, and so constituted as to render the different terms homogeneous, its roots may be as readily determined, by means of the simple substitution of a new unknown quantity, as if they were entirely numeral.

Thus, let the equation be

$$x^4+4ax^3-19a^2x^2+4a^3x+a^4=0,$$

where the given and unknown quantities can be substituted, alternately, for each other, without changing its form.

Then, by putting $x=az$, we shall have

$$a^4z^4+4a^4z^3-19a^4z^2+4a^4z+a^4=0,$$

or (by dividing by a^4) $z^4+4z^3-19z^2+4z+1=0$; which, according to the first equation given above, will have for its roots

$$\frac{3}{2}+\frac{1}{2}\sqrt{5}, \frac{3}{2}-\frac{1}{2}\sqrt{5}, -\frac{7}{2}+\frac{3}{2}\sqrt{5}, \text{ and } -\frac{7}{2}-\frac{3}{2}\sqrt{5};$$

and consequently the four roots of the proposed equation are

$$a\left(\frac{3}{2}+\frac{1}{2}\sqrt{5}\right), a\left(\frac{3}{2}-\frac{1}{2}\sqrt{5}\right), a\left(-\frac{7}{2}+\frac{3}{2}\sqrt{5}\right), \,\&\, a\left(-\frac{7}{2}-\frac{3}{2}\sqrt{5}\right).$$

3. Given $x^4 - 15x^3 + 38x^2 - 15x + 1 = 0$, to find the four roots of the equation.

Ans. $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$, and $6 \pm \sqrt{143}$

4. Given $x^4 - 4ax^3 + 5a^2x^2 - 4a^3x + a^4 = 0$, to find the four roots of the equation.

Ans. $a(\frac{3}{2} \pm \frac{1}{2}\sqrt{5})$, and $a(\frac{1}{2} \pm \frac{1}{2}\sqrt{-3})$

5. Given $x^5 - 21x^4 + 37x^3 + 37x^2 - 21x + 1 = 0$, to find the five roots of the equation.

Ans. -1, $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$, and $\frac{19}{2} \pm \frac{1}{2}\sqrt{357}$

6. Given $7x^6 - 26x^5 - 26x + 7 = 0$, to find the roots of the equation.

Ans. The only real roots are $2 \pm \sqrt{3}$

7. Given $2x^7 - 13a^2x^5 - 13a^5x^2 + 2a^7 = 0$, to find the roots of the equation.

Ans. The only real roots are $-a$, $a(\frac{3}{2} \pm \frac{1}{2}\sqrt{5})$, and

$$-\frac{a}{4}\{(2 + \sqrt{14}) \pm \sqrt{(2 + 4\sqrt{14})}\}.$$

OF BINOMIAL EQUATIONS, OR SUCH AS ARE OF THE FORM

$$x^m \pm a^m = 0.$$

(v) EQUATIONS of this kind may be reduced to a more simple form, by putting $x = ay$, and then dividing the result by a^m; in which case we shall have

$$a^m y^m \pm a^m = 0, \text{ or } y^m \pm 1 = 0,$$

where the several values of y are the roots of -1, or $+1$; and consequently those of x are the same roots multiplied by a.

Whence, as the first of these forms, $x^m + 1 = 0$, (using x instead of y,) is a reciprocal equation, wanting all the intermediate terms, its solution may be readily obtained from the rule given in the last article, by making the coefficients, p, q, r, s, each equal to 0, and finding the result accordingly.

It therefore only remains to determine the several roots of the equation

$$x^m - 1 = 0, \text{ or } x^m = 1,$$

which, for the first ten orders, may be done as follows (*f*).

RULE.

When the index of the first term is an even number.

1. If the equation be of the 4th power, as $x^4 - 1 = 0$, where two of the roots are, evidently, 1 and -1, find the values of x in the equation $x^2 + 1 = 0$, and they will be the other two roots of the proposed equation.

2. If it be of the 6th power, or $x^6 - 1 = 0$, where two of the roots are 1 and -1, as before, find the

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(*f*) The circumstance of unity, in the equations here treated of, having a number of roots besides 1, appears, at first, something singular; but, though the expressions, by which these roots are denoted, are, for the most part, imaginary, and for that reason might be thought of no importance, they are, notwithstanding, of great use in many of the higher branches of analysis, particularly in what is frequently called the arithmetic of sines, where the employment of symbols of this kind leads to many curious and interesting properties of the circle, and other curves, which could not be easily derived in any other way.

It appears, from the above rule, that, in the present state of Algebra, we are able to find all the roots of the equation values of x in each of the equations $x^2+x+1=0$, and $x^2-x+1=0$, and they will be the other four roots of the proposed equation.

3. If it be of the 8th power, or $x^8-1=0$, where two of the roots are, also, 1 and -1, find the two values of z in the equation $z^2-2=0$, or $z^2=2$, and let them be denoted by r, and r'; then the roots of the three quadratic equations $x^2+1=0$, $x^2-rx+1=0$, and $x^2-r'x+1=0$, will be the other six roots of the proposed equation.

4. And if it be of the 10th power, as $x^{10}-1=0$, where two of the roots are 1 and -1, as they are in all even powers, find the four roots of the equation $z^4-3z^2+1=0$, and let them be denoted by r, r', r'', r'''; then the roots of the four quadratic equations $x^2-rx+1=0$, $x^2-r'x+1=0$, $x^2-r''x+1=0$, and $x^2-r'''x+1=0$, will be the other eight roots of the proposed equation.

When the index of the first term is an odd number.

1. If the equation be of the 3d power, as $x^3-1=0$, where one of its roots is, evidently, 1,

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$x^m-1=0$, when m is not greater than 10, as the reduced equation, in that case, never exceeds the 4th order; but the next higher equation, $x^{11}-1=0$, is irresolvable, as the solution of it leads to an equation of the 5th power, the roots of which cannot be generally obtained.

But, although the algebraical calculus is confined within the limits here mentioned, the roots of any binomial equation whatever, of the kind

$$x^m \pm 1=0, \text{ or } x^m \pm a^m=0,$$

can be expressed, universally, by means of the division of the circumference of the circle into m equal parts; as is shown in the corresponding article, Vol. II, of the present work.

find the values of x in the quadratic equation $x^2 + x + 1 = 0$, and they will be the other two roots of the proposed equation.

2. If it be of the 5th power, or $x^5 - 1 = 0$, where one of its roots is 1, as before, find the two values of z in the equation $z^2 + z - 1 = 0$, and let them be denoted by r and r'; then the roots of the two quadratic equations $x^2 - rx + 1 = 0$, and $x^2 - r'x + 1 = 0$, will be the other four roots of the proposed equation.

3. If it be of the 7th power, or $x^7 - 1 = 0$, where one of its roots is also 1, find the three values of z in the equation $z^3 + z^2 - 2z - 1 = 0$, and let them be denoted by r, r'', r'''; then the roots of the three quadratic equations $x^2 - rx + 1 = 0$, $x^2 - r'x + 1 = 0$, and $x^2 - r''x + 1 = 0$, will be the other six roots of the proposed equation.

4. And if it be of the 9th power, as $x^9 - 1 = 0$, where one of its roots is likewise 1, as it is in all the odd powers, find the four values of z in the equation $z^4 + z^3 - 3z^2 - 2z + 1 = 0$, and let them be denoted by r, r', r'', r'''; then the roots of the four quadratic equations $x^2 - rx + 1 = 0$, $x^2 - r'x + 1 = 0$, $x^2 - r''x + 1 = 0$, and $x^2 - r'''x + 1 = 0$, will be the other eight roots of the proposed equation.

To this we may also add, that, since two of the values of x, in every case of even powers, are $+1$ and -1, if the first member of an equation of the kind $x^{2m} - 1 = 0$, be divided by $x^2 - 1$, the quotient will be

$$x^{2m-2} + x^{2m-4} + x^{2m-6} + \ldots\ldots x^4 + x^2 + 1 = 0.$$

And because one of the values of x, in every case of the odd powers, is 1, if the first member of any equation of the kind $x^{2n+1}-1=0$, be divided by $x-1$, the quotient will be

$$x^{2n}+x^{2n-1}+x^{2n-2}+x^{2n-3}+\ldots\ldots+x^2+x+1=0.$$

Both of which are reciprocal equations, having all the coefficients of their several terms $=1$, and consequently each of them is resolvable by the method employed in the last article.

EXAMPLES.

1. It is required to find the three roots of the equation $x^3-1=0$.

Here, the index of x being an odd number, one of the roots is 1.

And, by the second part of the rule,

$$x^2+x+1=0, \text{ or } x^2+x=-1.$$

Whence, by quadratics,

$$x=-\tfrac{1}{2}\pm\sqrt{(\tfrac{1}{4}-1)}=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$

Consequently,

$$1,\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}, \text{ and } -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3},$$

are the roots sought.

And if it were required to find the three roots of the equation $x^3+1=0$,

We shall have, in this case, one of the roots $=-1$,

And, by dividing x^3+1 by $x+1$, there will arise

$$x^2-x+1=0, \text{ or } x^2-x=-1.$$

Whence, as before,

$$x=\tfrac{1}{2}\pm\sqrt{(\tfrac{1}{4}-1)}=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$

Therefore,

$$-1,\ \tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}, \text{ and } \tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3},$$

are the roots sought.

2. It is required to find the five roots of the equation $x^5-1=0$.

Here 5 being an odd power, one root of the given equation is $=1$.

Therefore, by the rule,

$$z^2+z-1=0, \text{ or } z^2+z=1, \text{ where}$$

$$z=-\tfrac{1}{2}+\sqrt{(\tfrac{1}{4}+1)}=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}=r, \text{ and}$$

$$z=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{5}=r'.$$

Whence, also,

$$x^2-rx+1=0, \text{ or } x^2-(-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5})x=-1.$$

And consequently,

$$x=\tfrac{1}{4}(-1+\sqrt{5})\pm\sqrt{\{\tfrac{1}{16}(-1+\sqrt{5})^2-1\}}=$$

$$\tfrac{1}{4}(-1+\sqrt{5})\pm\tfrac{1}{4}\sqrt{\overline{-10-2\sqrt{5}}}=\tfrac{1}{4}(-1+\sqrt{5})\pm$$

$$\tfrac{1}{4}(\sqrt{\overline{10+2\sqrt{5}}})\sqrt{-1}.$$

In like manner,

$$x^2-r'x+1=0, \text{ or } x^2+(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5})x=-1.$$

Where $x=-\tfrac{1}{4}(1+\sqrt{5})\pm\sqrt{\{\tfrac{1}{16}(1+\sqrt{5})^2-1\}}=-$

$$\tfrac{1}{4}(1+\sqrt{5})\pm\tfrac{1}{4}(\sqrt{\overline{10-2\sqrt{5}}})\sqrt{-1}.$$

Therefore the five roots are

$$1,\ \tfrac{1}{4}(-1+\sqrt{5})\pm\tfrac{1}{4}(\sqrt{\overline{10+2\sqrt{5}}})\sqrt{-1}, \text{ and}$$

$$-\tfrac{1}{4}(1+\sqrt{5})\pm\tfrac{1}{4}(\sqrt{\overline{10-2\sqrt{5}}})\sqrt{-1}.$$

3. It is required to find the four roots of the equation $x^4-1=0$.

Ans. $+1$, -1, $+\sqrt{-1}$, and $-\sqrt{-1}$

4. It is required to find the five roots of the equation $x^5+1=0$.

Ans. -1, $\tfrac{1}{4}(1+\sqrt{5})\pm\tfrac{1}{4}(\sqrt{\overline{10-2\sqrt{5}}})\sqrt{-1}$, and $\tfrac{1}{4}(1-\sqrt{5})\pm\tfrac{1}{4}(\sqrt{\overline{10+2\sqrt{5}}})\sqrt{-1}$

5. It is required to find the six roots of the equation $x^6 - a^6 = 0$.

Ans. a, $-a$, $a(-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3})$, and $a(\frac{1}{2} \pm \frac{1}{2}\sqrt{-3})$

6. It is required to find the eight roots of the equation $x^8 - a^8 = 0$.

Ans. a, $-a$, $a\sqrt{-1}$, $-a\sqrt{-1}$, $a(\frac{1}{2}\sqrt{2} \pm \frac{1}{2}\sqrt{-2})$, and $a(-\frac{1}{2}\sqrt{2} \pm \frac{1}{2}\sqrt{-2})$

7. It is required to find the ten roots of the equation $x^{10} - a^{10} = 0$.

OF EQUATIONS THAT HAVE EQUAL ROOTS.

(U) Besides the several classes of equations, before treated of, there are others of a different kind, that are equally susceptible of being reduced to those of lower dimensions, the most useful of them being such as have two or more equal roots, with the same or contrary signs; in which case the methods of resolving them, as far as to the 4th order, inclusively, are as follows:

Rule.

1. If a quadratic equation, of the general form $x^2 + ax + b = 0$, have two equal roots, with the same sign, they will be each $= -\frac{1}{2}a$. And if the equal roots have contrary signs, the equation must be of the form $x^2 - b^2 = 0$; in which case, $x = \pm\sqrt{b^2}$, or $+b$ and $-b$.

2. If the cubic equation $x^3 + ax^2 + bx + c = 0$, has two equal roots with the same sign, each of them will be a root of the equation $3x^2 + 2ax + b = 0$, and the remaining root will be $= -a -$ twice one of the equal roots. And, if they have contrary signs, we

shall have, in that case, $ab = c$, and the roots sought will be $\sqrt{-\frac{c}{a}}$, and $-\sqrt{-\frac{c}{a}}$, and the remaining root will be $= a$.

3. If a biquadratic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

has two equal roots, with the same sign, each of them will be a root of the equation

$$(3a^2 - 8b)x^2 + 2(ab - 6c)x + ac - 16d = 0;$$

and the other two roots will be found from the equation

$$x^2 + (a + 2r)x + b + 2ar + 3r^2 = 0,$$

r being one of the equal roots before found. And if the equal roots have contrary signs, we shall have $a(bc - ad) = c^2$, and $x = \pm\sqrt{-\frac{c}{a}}$, as before; the other two roots being found from the equation

$$x^2 + ax + b + r^2 = 0.$$

4. Also, if the biquadratic

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

has three equal roots, with the same sign, each of them will be a root of the equation $6x^2 + 3ax = -b$. And if two of the three equal roots have contrary signs, we shall have $a(bc - ad) = c^2$, and $x = \pm\sqrt{-\frac{c}{a}}$ as in the last case (g).

(g) The mode of solution, here given, might have been easily extended to equations of the 5th, 6th, &c. power; but in those above the 4th degree, the expressions for the equal roots, when they have the same sign, are too complicated to be of any practical use. Rules might also have been introduced for determining *a priori*, by means of the relations of the coefficients alone, whether or not any given equation has equal roots; but, except in the case where they have contrary signs, this may, in general,

EXAMPLES.

1. Supposing that two of the roots of the equation $x^3+x^2-33x+63=0$, are equal to each other, and have the same sign, it is required to determine them.

Here a being $=1$, and $b=-33$, we shall have, by the rule,

$$3x^2+2ax+b=3x^2+2x-33=0;\ \text{or}\ x^2+\frac{2}{3}x=11.$$

Whence

$$x=-\frac{1}{3}\pm\sqrt{(\frac{1}{9}+11)}=-\frac{1}{3}\pm\sqrt{\frac{100}{9}}=-\frac{1}{3}\pm\frac{10}{3}=3,$$

$$\text{or}\ -\frac{11}{3}$$

The former of which (3) being substituted for x, in the original equation, is found to succeed.

Whence, two of the three roots are, each $=3$; and the remaining root $=-7$.

2. Supposing that two of the roots of the equation

$$x^4+\frac{1}{2}x^3-\frac{3}{2}x^2+\frac{4}{27}x+\frac{2}{9}=0,$$

are equal to each other, and have the same sign; it is required to determine them.

Here a being $=\frac{1}{2}$, $b=-\frac{3}{2}$, $c=\frac{4}{27}$, and $d=\frac{2}{9}$, we shall have, by the rule,

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be more readily done, by finding the root of the quadratic equation to which it is reducible, and then seeing whether the root, thus obtained, be a root of the original equation.

In all cases, of this kind, however, it may be of use to know, that an equation cannot have equal roots when the last term and the coefficient of the last but one, are prime to each other; but the reverse of this is not always true.

$$(3a^2-8b)x^2+2(ab-6c)x+ac-16d=\frac{51}{4}x^2-\frac{59}{18}x-\frac{94}{27}=0.$$

Or, by transposition and simplifying the terms,

$$x^2-\frac{118}{459}x=\frac{376}{1377};$$

The two roots of which equation are $\frac{2}{3}$ and $-\frac{188}{459}$

And, by trial, the first of these, $\frac{2}{3}$, will be found to be a root of the original equation proposed; whence the two roots required are each $=\frac{2}{3}$

And, since $r=\frac{2}{3}$, we shall have, by the rule,

$$x^2+(a+2r)x+b+2ar+3r^2=x^2+$$
$$(\frac{1}{2}+\frac{4}{3})x-\frac{3}{2}+\frac{2}{3}+\frac{4}{3}=0, \text{ or } x^2+\frac{11}{6}x=-\frac{1}{2};$$

the roots of which latter equation are $-\frac{1}{3}$, and $-\frac{3}{2}$

Whence the four roots of the given equation are

$$\frac{2}{3},\ \frac{2}{3}\ -\frac{1}{3}, \text{ and } -\frac{3}{2}$$

3. It is required to determine whether the equation

$$x^4+11x^3+19x^2-99x-252=0,$$

has equal roots, and, if so, what they are.

Here, since

$$a=11,\ b=19,\ c=-99, \text{ and } d=-252,$$

we shall have, by the rule,

$$a(bc-ad)=11(-19\times 99+11\times 252)=$$
$$11(-1881+2772)=9801=99^2=c^2;$$

whence the equation has two equal roots, with contrary signs.

And consequently $x=\pm\sqrt{-\frac{c}{a}}=\pm\sqrt{\frac{99}{11}}=\pm\sqrt{9}=3$

and -3; which two roots will be found to answer the conditions of the question.

Also, since $r = \pm 3$, we shall have, by the rule,

$$x^2 + ax + b + r^2 = x^2 + 11x + 28 = 0,$$

$$\text{or } x^2 + 11x = -28;$$

The roots of which equation are -4 and -7, Whence the four roots of the given equation are

$$3,\ -3,\ -4,\ \text{and } -7.$$

4. Supposing that two roots of the equation $x^3 - 46x - 128 = 0$, are equal, and have contrary signs; it is required to determine them.

Ans. -4, -4, and 8

5. Given the equation $x^3 + 5\frac{1}{2}x^2 - \frac{1}{16}x - \frac{11}{32} = 0$, to find whether it has equal roots, and, if so, what they are. Ans. $\frac{1}{4}$, $-\frac{1}{4}$, and $-5\frac{1}{2}$

6. Supposing the equation $x^4 + 3x^3 - 14x^2 - 12x + 40 = 0$, to have three equal roots, two of which have the same sign; it is required to determine them.

Ans. 2, $2. -2$, and -5

7. Supposing the equation $x^4 - \frac{51}{400}x^2 - \frac{1}{800}x + \frac{3}{800} = 0$, to have two equal roots, with the same sign; it is required to determine all its four roots.

Ans. $-\frac{1}{4}$, $-\frac{1}{4}$, $\frac{1}{5}$, and $\frac{3}{10}$

OF EXPONENTIAL EQUATIONS.

(x) An exponential quantity is that which is to be raised to some unknown power, or which has a variable quantity for its index; as

$$a^x,\ a^{\frac{1}{x}},\ x^x,\ \text{or } x^{\frac{1}{x}},\ \&c.$$

And an exponential equation is that which is formed between any expression of this kind and some other quantity, whose value is known; as

$$a^x=b;\ x^x=a,\ \&c.$$

Where it is to be observed, that $a^x=b$, when converted into logarithms, is the same as $x\log.a=\log.b$; or $x=\frac{\log.b}{\log.a}$; and $x^x=a$ is the same as $x\log.x=\log.a$; in the latter of which cases, the value of the unknown quantity x may be determined, to any degree of exactness, by the method of double position, as follows:

RULE.

Find, by trial, two numbers as near the number sought as possible, and substitute them in the given equation $x\log.x=\log.a$, instead of the unknown quantity, noting the results obtained from each.

Then, as the difference of these results is to the difference of the two assumed numbers, so is the difference between the true result, given in the question, and either of the former, to the correction of the number belonging to the result used; which correction being added to that number, when it is too little, or subtracted from it, when it is too great, will give the root required, *nearly*.

And, if the number, thus determined, and the nearest of the two former, or any other that appears to be nearer, be taken as the assumed roots, and the operation be repeated as before, a new value of the unknown quantity will be obtained still more correct than the first; and so on, proceeding in this manner, as far as may be thought necessary (*h*).

EXAMPLES.

1. Given $x^x = 100$, to find an approximate value of x.

Here, by the above formula, we have

$$x \log. x = \log. 100 = 2,$$

And since x is readily found, by a few trials, to be nearly in the middle between 3 and 4, but rather nearer the latter than the former, let 3.5 and 3.6 be taken for the two assumed numbers.

Then log. 3.5 = .5440680, which, being multiplied by 3.5, gives 1.904238 = first result;

And log. 36 = .5563025, which, being multiplied by 3.6, gives 2.002689 for the second result.

Whence

$$\begin{array}{ccccccc} 2.002689 & & 3.6 & & 2.002689 & & \\ 1.904238 & & 3.5 & & 2. & & \\ \hline .098451 & : & .1 & :: & .002689 & . & .00273 \end{array}$$

For the first correction; which, taken from 3.6, leaves $x = 3.59727$, *nearly*.

And as this value is found, from trial, to be rather too small, let 3.59727 and 3.59728 be taken as the two assumed numbers.

Then log. 3.59727 = .5559731, which, being multiplied by 3.59727, gives 1.9999854 = first result.

(*h*) Functions of the kind here treated of were first introduced into some of the higher branches of analysis by John Bernoulli, who published a paper on this subject, in the Leipsic Acts for 1697; but the denomination of exponentials was given to them by Leibnitz.

And log. 3.59728 = .5559743, which, being multiplied by 3.59728, gives 1.9999953 = second result.

Whence

$$\begin{array}{lll} 1.9999953 & 3.59728 & 2. \\ 1.9999854 & 3.59727 & 1.9999953 \\ \hline .0000099 : & .00001 :: & .0000047 : .00000474747 \end{array}$$

For the second correction; which, added to 3.59728, gives $x = 3.59728474747$; *extremely near the truth.*

2. Given $x^x = 2000$, to find an approximate value of x. Ans. $x = 4.82782263$

3. Given $(6x)^x = 96$, to find the approximate value of x. Ans. $x = 1.8826432$

4. Given $x^x = 123456789$, to find the value of x. Ans. 8.6400268

5. Given $x^x - x = (2x - x^x)^{\frac{1}{x}}$, to find the value of x. Ans. $x = 1.747933$

OF THE BINOMIAL THEOREM.

(Y) THE binomial theorem is a general algebraical expression, or formula, by which any power, or root of a given quantity, consisting of two terms, is expanded into a series; the form of which, as it was first proposed by Newton, being as follows:

$$(\text{P} + \text{PQ})^{\frac{m}{n}} = \text{P}^{\frac{m}{n}}[1 + \frac{m}{n}\text{Q} + \frac{m}{n}(\frac{m-n}{2n})\text{Q}^2 + \frac{m}{n}(\frac{m-n}{2n})(\frac{m-2n}{3n})\text{Q}^3 \&\text{c.}]$$

Or

$$(\text{P} + \text{PQ})^{\frac{m}{n}} = \text{P}^{\frac{m}{n}} + \frac{m}{n}\text{AQ} + \frac{m-n}{2n}\text{BQ} + \frac{m-2n}{3n}\text{CQ} + \frac{m-3n}{4n}\text{DQ} \&\text{c.}$$

Where P is the first term of the binomial, Q the second term divided by the first, $\frac{m}{n}$ the index of the power, or root, and A, B, C, &c. the terms immediately preceding those in which they are first found, including their signs + or −.

Which theorem may be readily applied to any particular case, by substituting the numbers, or letters, in the given example, for P, Q, m, and n, in either of the above formulæ, and then finding the result according to the rule (*i*).

EXAMPLES.

1. It is required to convert $(a^2+x)^{\frac{1}{2}}$ into an infinite series.

Here $P=a^2$, $Q=\frac{x}{a^2}$, $\frac{m}{n}=\frac{1}{2}$, or $m=1$, and $n=2$; whence

$$P^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{\frac{1}{2}}=a=A,$$

$$\frac{m}{n}AB=\frac{1}{2}\times\frac{a}{1}\times\frac{x}{a^2}=\frac{x}{2a}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{x}{2a}\times\frac{x}{a^2}=-\frac{x^2}{2.4a^3}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times-\frac{x^2}{2.4a^3}\times\frac{x}{a^2}=\frac{3x^3}{2.4.6a^5}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{3x^3}{2.4.6a^5}\times\frac{x}{a^2}=-\frac{3.5x^4}{2.4.6.8a^7}=E,$$

$$\frac{m-4n}{5n}EQ=\frac{1-8}{10}\times-\frac{3.5x^4}{2.4.6.8a^7}\times\frac{x}{a^2}=\frac{3.5.7x^5}{2.4.6.8.10a^9}=F,$$

&c. &c.

(*i*) This celebrated theorem, which is of the most extensive use in algebra, and various other branches of analysis, may be otherwise expressed as follows:

$$(a+x)^{\frac{m}{n}}=a^{\frac{m}{n}}[1+\frac{m}{n}(\frac{x}{a})+\frac{m}{n}.\frac{m-n}{2n}(\frac{x}{a})^2+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}(\frac{x}{a})^3\,\&c.],$$

Or $(a+x)^{\frac{m}{n}}=$

$$a^{\frac{m}{n}}[1+\frac{m}{n}(\frac{x}{a+x})+\frac{m}{n}.\frac{m+n}{2n}(\frac{x}{a+x})^2+\frac{m}{n}.\frac{m+n}{2n}.\frac{m+2n}{3n}(\frac{x}{a+x})^3\&c.]$$

Therefore $(a^2+x)^{\frac{1}{2}} =$

$$a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}+\frac{3.5.7x^5}{2.4.6.8.10a^9}-\&c.$$

Where the law of formation of the several terms of the series is sufficiently evident.

2. It is required to find the value of $\frac{1}{(a+b)^2}$, or its equal $(a+b)^{-2}$, in an infinite series.

Here $P=a$, $Q=\frac{b}{a}$, and $\frac{m}{n}=-2$, or $m=-2$ and $n=1$; whence

$$P^{\frac{m}{n}}=(a)^{\frac{m}{n}}=a^{-2}=\frac{1}{a^2}=A,$$

$$\frac{m}{n}AQ=-\frac{2}{1}\times\frac{1}{a^2}\times\frac{b}{a}=-\frac{2b}{a^3}=B,$$

$$\frac{m-n}{2n}BQ=\frac{-2-1}{2}\times-\frac{2b}{a^3}\times\frac{b}{a}=\frac{3b^2}{a^4}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{-2-2}{3}\times\frac{3b^2}{a^4}\times\frac{b}{a}=-\frac{4b^3}{a^5}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{-2-3}{4}\times-\frac{4b^3}{a^5}\times\frac{b}{a}=\frac{5b^4}{a^6}=E,$$

&c. &c.

Or $(a+x)^{\frac{m}{n}} =$

$$(2a)^{\frac{m}{n}}[1-\frac{m}{n}(\frac{a-x}{a+x})+\frac{m}{n}.\frac{m+n}{2n}(\frac{a-x}{a+x})^2-\frac{m}{n}.\frac{m+n}{2n}.\frac{m+2n}{3n}(\frac{a-x}{a+x})^3\&c.]$$

And if the reciprocals of the same expressions be required, they will be

$$\frac{1}{(a+x)^{\frac{m}{n}}}=\frac{1}{a^{\frac{m}{n}}}[1-\frac{m}{n}(\frac{x}{a})+\frac{m}{n}.\frac{m+n}{2n}(\frac{x}{a})^2-\frac{m}{n}.\frac{m+n}{2n}.\frac{m+2n}{3n}(\frac{x}{a})^3\&c.]$$

Or $\frac{1}{(a+x)^{\frac{m}{n}}}=$

$$(\frac{1}{a})^{\frac{m}{n}}[1-\frac{m}{n}(\frac{x}{a+x})+\frac{m}{n}.\frac{m-n}{2n}(\frac{x}{a+x})^2-\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}(\frac{x}{a+x})^3\ \&c.]$$

Consequently $\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}$&c.

3. It is required to find the value of $\frac{a^2}{(a^2-x)^{\frac{1}{2}}}$, or its equal $a^2(a^2-x)^{-\frac{1}{2}}$, in an infinite series.

Here

$\text{P}=a^2$, $\text{Q}=-\frac{x}{a^2}$, and $\frac{m}{n}=\frac{-1}{2}$, or $m=-1$ and $n=2$;

whence

$$\text{P}^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{-\frac{1}{2}}=\frac{1}{a}=\text{A},$$

$$\frac{m}{n}\text{AQ}=-\frac{1}{2}\times\frac{1}{a}\times-\frac{x}{a^2}=\frac{x}{2a^3}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{-1-2}{4}\times\frac{x}{2a^3}\times-\frac{x}{a^2}=\frac{3x^2}{2.4a^5}=\text{C},$$

$$\frac{m-2n}{3n}\text{CQ}=\frac{-1-4}{6}\times\frac{3x^2}{2.4a^5}\times-\frac{x}{a^2}=\frac{3.5x^3}{2.4.6a^7}=\text{D},$$

$$\frac{m-3n}{4n}\text{DQ}=\frac{-1-6}{8}\times\frac{3.5x^3}{2.4.6a^7}\times-\frac{x}{a^2}=\frac{3.5.7x^4}{2.4.6.8a^9}=\text{E},$$

&c. &c.

$$\text{Or }\frac{1}{(a+x)^{\frac{m}{n}}}=$$

$$\left(\frac{1}{2a}\right)^{\frac{m}{n}}\left[1+\frac{m}{n}\left(\frac{a-x}{a+x}\right)+\frac{m}{n}.\frac{m-n}{2n}\left(\frac{a-x}{a+x}\right)^2+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}\left(\frac{a-x}{a+x}\right)^3\text{ \&c.}\right]$$

It may here also be observed, that if m be made to represent any whole, or fractional number, whether positive or negative, the first of these expressions may be exhibited in the more simple form

$$(a+x)^m=a^m+ma^{m-1}x+\frac{m(m-1)}{1\ .\ 2}a^{m-2}x^2+\frac{m(m-1)(m-2)}{1\ .\ 2\ .\ 3}a^{m-3}x^3.$$

$$\ldots\ldots\ \frac{m(m-1)(m-2)\ .\ .\ .\ .\ .\ [m-(n-1)]a^nx^{m-n}}{1\ .\ 2\ .\ 3\ .\ 4\ .\ .\ .\ .\ .\ .\qquad n}$$

Where the last term is called the *general term of the series*, because if 1, 2, 3, 4, &c. be substituted successively for n, it will give all the rest.

Therefore

$$\frac{1}{(a^2-x)^{\frac{1}{2}}}=\frac{1}{a}+\frac{1}{2}\left(\frac{x}{a^3}\right)+\frac{3}{2.4}\left(\frac{x^2}{a^5}\right)+\frac{3.5}{2.4.6}\left(\frac{x^3}{a^7}\right)+\frac{3.5.7}{2.4.6.8}\left(\frac{x^4}{a^9}\right)\&c.$$

And

$$\frac{a^2}{(a^2-x)^{\frac{1}{2}}}=a+\frac{1}{2}\left(\frac{x}{a}\right)+\frac{3}{2.4}\left(\frac{x^2}{a^3}\right)+\frac{3.5}{2.4.6}\left(\frac{x^3}{a^5}\right)+\frac{3.5.7}{2.4.6.8}\left(\frac{x^4}{a^7}\right)\&c.$$

4. It is required to find the value of $(8+1)^{\frac{1}{3}}$, or $\sqrt[3]{9}$, in an infinite series.

Here $P=8$, $Q=\frac{1}{8}$, and $\frac{m}{n}=\frac{1}{3}$, or $m=1$ and $n=3$; whence

$$P^{\frac{m}{n}}=(8)^{\frac{m}{n}}=8^{\frac{1}{3}}=2=A,$$

$$\frac{m}{n}AQ=\frac{1}{3}\times\frac{2}{1}\times\frac{1}{2^3}=\frac{1}{3.2^2}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-3}{6}\times\frac{1}{3.2^2}\times\frac{1}{2^3}=-\frac{1}{3.6.2^4}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-6}{9}\times-\frac{1}{3.6.2^4}\times\frac{1}{2^3}=\frac{5}{3.6.9.2^7}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{1-9}{12}\times\frac{5}{3.6.9.2^7}\times\frac{1}{2^3}=-\frac{5.8}{3.6.9.12.2^{10}}=E,$$

$$\frac{m-4n}{5n}EQ=\frac{1-12}{15}\times-\frac{5.8}{3.6.9.12.2^{10}}\times\frac{1}{2^3}=\frac{5.8.11}{3.6.9.12.15.2^{13}}=F,$$

&c. &c.

Therefore $\sqrt[3]{9}=$

$$2+\frac{1}{3.2^2}-\frac{1}{3.6.2^4}+\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}+\frac{5.8.11}{3.6.9.12.15.2^{13}}\&c.$$

5. It is required to find the value of $\sqrt{2}$, or $\sqrt{(1+1)}$, in an infinite series.

$$\text{Ans. } 1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\frac{3.5.7}{2.4.6.8.10}\&c.$$

6. It is required to find the value of $(8-1)^{\frac{1}{3}}$, or $\sqrt{7}$, in an infinite series.

$$\text{Ans. } 2-\frac{1}{3.2^2}-\frac{1}{3.6.2^4}-\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}-\frac{5.8.11}{3.6.9.12.15.2^{13}}\&c.$$

7. It is required to find the value of $(a \pm x)^{\frac{1}{2}}$, in an infinite series.

$$\text{Ans. } a^{\frac{1}{2}}\{1 \pm \frac{x}{2a} - \frac{x^2}{2.4a^2} \pm \frac{3x^3}{2.4.6a^3} - \frac{3.5x^4}{2.4.6.8a^4} \pm \&c.\}$$

8. It is required to find the value of $(a \pm b)^{\frac{1}{3}}$, in an infinite series.

$$\text{Ans. } a^{\frac{1}{3}}\{1 \pm \frac{b}{3a} - \frac{2b^2}{3.6a^2} \pm \frac{2.5b^3}{3.6.9a^3} - \frac{2.5.8b^4}{3.6.9.12a^4} \pm \&c.\}$$

9. It is required to find the value of $(a - b)^{\frac{1}{4}}$, in an infinite series.

$$\text{Ans. } a^{\frac{1}{4}}\{1 - \frac{b}{4a} - \frac{3b^2}{4.8a^2} - \frac{3.7b^3}{4.8.12a^3} - \frac{3.7.11b^4}{4.8.12.16a^4} - \&c.\}$$

10. It is required to find the value of $(a + x)^{\frac{2}{3}}$, in an infinite series.

$$\text{Ans. } a^{\frac{2}{3}}\{1 + \frac{2x}{3a} - \frac{x^2}{9a^2} + \frac{4x^3}{9^2a^3} - \frac{4.7x^4}{9^2 12a^4} + \frac{4.7.10x^5}{9^2.12.15a^5} - \&c.\}$$

11. It is required to find the value of $\frac{1}{(a \pm x)^{\frac{1}{2}}}$, in an infinite series.

$$\text{Ans. } \frac{1}{a^{\frac{1}{2}}}\{1 \mp \frac{x}{2a} + \frac{3x^2}{2.4a^2} \mp \frac{3.5x^3}{2.4.6a^3} + \frac{3.5.7x^4}{2.4.6.8a^4} \mp \&c.\}$$

12. It is required to find the value of $\frac{a}{(a \pm x)^{\frac{1}{3}}}$, in an infinite series.

$$\text{Ans. } a^{\frac{2}{3}}\{1 \mp \frac{x}{3a} + \frac{2x^2}{3.6a^2} \mp \frac{2.5x^3}{3.6.9a^3} + \frac{2.5.8x^4}{3.6.9.12a^4} \mp \&c.\}$$

13. It is required to find the value of $(1 - x)^{\frac{2}{5}}$ in an infinite series.

$$\text{Ans. } 1 - \frac{2x}{5} - \frac{2.3x^2}{5.10} - \frac{2.3.8x^3}{5.10.15} - \frac{2.3.8.13x^4}{5.10.15.20} \&c.$$

14. It is required to find the value of $\frac{1}{(1 + x)^{\frac{1}{5}}}$ in an infinite series.

$$\text{Ans. } 1 - \frac{x}{5} + \frac{6x^2}{5.10} - \frac{6.9x^3}{5.10.15} + \frac{6.9.14x^4}{5.10.15.20} \&c.$$

15. It is required to find the value of $(\frac{a+x}{a-x})^{\frac{1}{2}}$, in an infinite series.

$$\text{Ans. } 1+\frac{x}{a}+\frac{x^2}{2a^2}+\frac{x^3}{2a^3}+\frac{3x^4}{8a^4}+\frac{3x^5}{8a^5}+\frac{5x^6}{16a^6}+\frac{5x^7}{16a^7}, \text{ \&c.}$$

OF THE MULTIPLICATION AND DIVISION OF SERIES.

(z) ANY two series may be multiplied or divided by each other, by substituting the numbers, or letters, of the given example, in the place of their corresponding terms, in the following general formulæ, and then finding the result accordingly (*k*).

$$\begin{array}{l}
A_0+A_1x+A_2x^2+A_3x^3+A_4x^4\text{\&c.}\\
B_0+B_1x+B_2x^2+B_3x^3+B_4x^4\text{\&c.}\\
\hline
\end{array}$$

$$\text{Prod}^t = \begin{array}{|l l|l l|l l|l l|l}
A_0B_0 & +A_1B_0 & x & +A_2B_0 & x^2 & +A_3B_0 & x^3 & +A_4B_0 & x^4\ \text{\&c.}\\
 & +A_0B_1 & & +A_1B_1 & & +A_2B_1 & & +A_3B_1 & \\
 & & & +A_0B_2 & & +A_1B_2 & & +A_2B_2 & \\
 & & & & & +A_0B_3 & & +A_1B_3 & \\
 & & & & & & & +A_0B_4 &
\end{array}$$

And if $B_0=A_0$, $B_1=A_1$, &c. the formula for the square of the upper, or following, series,

$$A_0+A_1x+A_2x^2+A_3x^3+A_4x^4\text{\&c.}$$

will, in that case, be as follows:

(*k*) In the mode of notation here followed, it is to be observed, that A_0, A_1, A_2, are the coefficients of the several powers of x, so taken, that the inferior figures 0, 1, 2, 3, &c. by which they are denoted, shall always agree with the indices of the unknown quantities that are attached to them; by which means, the law of the continuation of the terms is, in all cases, rendered obvious, and a conformity between the coefficients and powers of x constantly preserved.

$$\text{Square} = \left|\begin{array}{r|r|r|r|l} A_0^2 + A_1A_0 & x + A_2A_0 & x^2 + A_3A_0 & x^3 + A_4A_0 & x^4\,\&c. \\ + A_0A_1 & + A_1A_1 & + A_2A_1 & + A_3A_1 & \\ & + A_0A_2 & + A_1A_2 & + A_2A_2 & \\ & & + A_0A_3 & + A_1A_3 & \\ & & & + A_0A_4 & \end{array}\right.$$

$$\text{Also } \frac{A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4\,\&c.}{1 + B_1x + B_2x^2 + B_3x^3 + B_4x^4\,\&c.}$$

$$\text{Quo}^{t} = \left|\begin{array}{r|r|r|r|l} A_0 + A_1 & x + A_2 & x^2 + A_3 & x^3 + A_4 & x^4\,\&c. \\ - B_1C_0 & - B_2C_0 & - B_3C_0 & - B_4C_0 & \\ & - B_1C_1 & - B_2C_1 & - B_3C_1 & \\ & & - B_1C_2 & - B_2C_2 & \\ & & & - B_1C_3 & \end{array}\right.$$

And if A_0, in the upper series, be taken $= 1$, and the rest of the terms be put $= 0$, or made to vanish, we shall have, for the reciprocal of the lower series, or

$$\frac{1}{1 + B_1x + B_2x^2 + B_3x^3 + B_4x^4\,\&c.}$$

$$\text{Recip}^{l} = \left|\begin{array}{r|r|r|r|l} 1 - B_1C_0 & x - B_2C_0 & x^2 - B_3C_0 & x^3 - B_4C_0 & x^4\,\&c. \\ & - B_1C_1 & - B_2C_1 & - B_3C_1 & \\ & & - B_1C_2 & - B_2C_2 & \\ & & & - B_1C_3 & \end{array}\right.$$

Where it is to be observed, that, in each of the two latter series, C_0 denotes the first term, C_1 the coefficient of the second term, C_2 that of the third, and so on, according to the order C_3, C_4, &c. (*l*)

(*l*) The first terms of the denominators of the two last examples, given above, are, for the sake of simplicity, taken equal to unity; to which form any series can always be readily reduced; as is obvious from the following instance,

$$a + bx + cx^2 + dx^3 + ex^4\,\&c. = a\left(1 + \frac{b}{a}x + \frac{c}{a}x^2 + \frac{d}{a}x^3 + \frac{e}{x}x^4\right)\&c.$$

EXAMPLES.

1. It is required to find the product of the two series

$$1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4 \text{ and}$$

$$1-\frac{1}{3}x+\frac{1}{5}x^2-\frac{1}{7}x^3+\frac{1}{9}x^4-\frac{1}{11}x^5\&c.$$

$$\text{Here } A_0=1,\ A_1=\frac{1}{2},\ A_2=\frac{1}{3}\&c. \text{ and}$$

$$B_0=1,\ B_1=-\frac{1}{3},\ B_2=\frac{1}{5}\&c.$$

Whence

$$\begin{array}{r|r|r|r|l}
1+\frac{1}{2} & x+\frac{1}{3} & x^2+\frac{1}{4} & x^3+\frac{1}{5} & x^4\&c. \\
-\frac{1}{3} & -\frac{1}{6} & -\frac{1}{9} & -\frac{1}{12} & \\
 & +\frac{1}{5} & +\frac{1}{10} & +\frac{1}{15} & \\
 & & -\frac{1}{7} & -\frac{1}{14} & \\
 & & & +\frac{1}{9} & \\
\hline
\end{array}$$

$$1+\frac{1}{6}x+\frac{11}{30}x^2+\frac{121}{1260}x^3+\frac{281}{1260}x^4\&c. \text{ Ans.}$$

2. It is required to divide the series

$$1+\frac{1}{6}x+\frac{11}{30}x^2+\frac{121}{1260}x^3+\frac{281}{1260}x^4\&c.$$

by the series

$$1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4\&c.$$

$$\text{Here } A_0=1,\ A_1=\frac{1}{6},\ A_2=\frac{11}{30}\&c.;\ \text{and}$$

$$B_1=\frac{1}{2},\ B_2=\frac{1}{3},\ B_3=\frac{1}{4},\ \&c.$$

Whence

$$A_0 = 1 = C_0$$

$$A_1 - B_1C_0 = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3} = C_1$$

$$A_2 - B_2C_0 - B_1C_1 = \frac{11}{30} - \frac{1}{3} + \frac{1}{6} = \frac{1}{5} = C_2$$

$$A_3 - B_3C_0 - B_2C_1 - B_1C_2 = \frac{121}{1260} - \frac{1}{4} + \frac{1}{9} - \frac{1}{10} = -\frac{1}{7} = C_3$$

$$A_4 - B_4C_0 - B_3C_1 - B_2C_2 - B_1C_3 = \frac{281}{1260} - \frac{1}{5} + \frac{1}{12} - \frac{1}{15} + \frac{1}{14} = \frac{1}{9} = C_4 \&c.$$

Therefore, $1 - \frac{1}{3}x + \frac{1}{5}x^2 - \frac{1}{7}x^3 + \frac{1}{9}x^4 \&c. =$ quotient.

3. It is required to multiply $1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5 \&c.$ by $1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \frac{1}{5}x^4 + \frac{1}{6}x^5 \&c.$

Ans. $1 + x + \frac{5}{6}x^2 + \frac{2}{3}x^3 + \frac{8}{15}x^4 + \frac{13}{30}x^5 \&c.$

4. It is required to find the square of $1 - \frac{1}{2}x + \frac{2}{3}x^2 - \frac{3}{4}x^3 + \frac{4}{5}x^4 - \frac{5}{6}x^5 \&c.$

Ans. $1 - x + \frac{19}{12}x^2 - \frac{13}{6}x^3 + \frac{503}{180}x^4 - \frac{52}{15}x^5 \&c.$

5. It is required to divide $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 \&c.$ by $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \&c.$

Ans. $1 + x + \frac{1}{2}x^2 + \frac{5}{12}x^3 + \frac{7}{24}x^4 + \frac{191}{720}x^5 \&c.$

6. It is required to find the reciprocal of $1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \frac{1}{5}x^4 \&c.$

Ans. $1 - \frac{1}{2}x - \frac{1}{12}x^2 - \frac{1}{24}x^3 - \frac{19}{720}x^4 - \frac{27}{1440}x^5 \&c.$

OF THE MULTINOMIAL THEOREM.

(A) The multinomial theorem is a general expression, or formula, for determining any power or root of a given quantity, consisting of any number of terms; the form of which is as follows (*m*):

$$(A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 \dots\dots A_nx^n)^m = A_0^m +$$

$$\left.\begin{array}{r} mA_1B_0 \end{array}\right|\frac{x}{A_0} \left.\begin{array}{r} +\,2mA_2B_0 \\ +\,(m-1)A_1B_1 \end{array}\right|\frac{x^2}{2A_0} \left.\begin{array}{r} +\,3mA_3B_0 \\ +\,(2m-1)A_2B_1 \\ +\,(m-2)A_1B_2 \end{array}\right|\frac{x^3}{3A_0} \left.\begin{array}{r} +\,4mA_4B_0 \\ +\,(3m-1)A_3B_1 \\ +\,(2m-2)A_2B_2 \\ +\,(m-3)A_1B_3 \end{array}\right|\frac{x^4}{4A_0}\,\&c.$$

Where $B_0 = A_0^m$; and B_1, B_2, B_3, &c. are the coefficients of the terms immediately preceding those in which they first appear.

And the manner of applying the theorem to any particular case, is by substituting the numbers, or letters, in the given example, for A_0, A_1, A_2, &c. and *m*, as in the binomial theorem, before given.

EXAMPLES.

1. It is required to find the cube of the series $1 + x + x^2 + x^3 + x^4 + x^5$&c.

(*m*) This theorem was first given by De Moivre, N° 230, Phil. Trans. 1697, who also pointed out the law of its terms; but the formula was afterwards much simplified by Euler, in his *Calcul Différentiel*, c. 8, part ii; where it may be found, in a form nearly similar to that in which it is here proposed. Arbogast, also, in his work entitled *Du Calcul des Dérivations*, p. 12, et seq., has presented the same theorem under a very simple and commodious form; but the mode of notation, which he has employed for this purpose, is incompatible with the plan of the present performance.

Here $A_0 = 1$, $A_1 = 1$, $A_2 = 1$&c. and $m = 3$.

Whence

$$A_0{}^m = 1^m = 1 = B_0$$

$$mA_1B_0 = 3 \times 1 \times 1 = 3 = B_1$$

$$\frac{2mA_2B_0 + (m-1)A_1B_1}{2} = \frac{6 + 2 \times 3}{2} = 6 = B_2$$

$$\frac{3mA_3B_0 + (2m-1)A_2B_1 + (m-2)A_1B_2}{3} = \frac{9 + 15 + 6}{3} = 10 = B_3$$

$$\frac{4mA_4B_0 + (3m-1)A_3B_1 + (2m-2)A_2B_2 + (m-3)A_1B_3}{4} = 15 = B_4$$

&c. &c.

Therefore $1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5$&c. = required cube.

2. It is required to find the cube root of the series $1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \frac{1}{5}x^4 + \frac{1}{6}x^5$&c.

Here $A_0 = 1$, $A_1 = \frac{1}{2}$, $A_2 = \frac{1}{3}$, $A_3 = \frac{1}{4}$&c.; and $m = \frac{1}{3}$

Whence

$$A_0{}^m = 1^m = 1 = B_0$$

$$mA_1B_0 = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{6} = B_1$$

$$\frac{2mA_2B_0 + (m-1)A_1B_1}{2} = \frac{1}{2}\left(\frac{2}{9} - \frac{1}{18}\right) = \frac{1}{12} = B_2$$

$$\frac{3mA_3B_0 + (2m-1)A_2B_1 + (m-2)A_1B_2}{3} = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{54} - \frac{1}{72}\right) = \frac{35}{648} = B_3$$

$$\frac{4mA_4B_0 + (3m-1)A_3B_1 + (2m-2)A_2B_2 + (m-3)A_1B_3}{4} = \frac{433}{1944} = B_4$$

&c. &c.

Therefore $1 + \frac{1}{6}x + \frac{1}{12}x^2 + \frac{35}{648}x^3 + \frac{433}{1944}x^4$&c. Ans.

3. It is required to find the square of the series $1 - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9$&c.

Ans. $1 - \frac{2}{3}x^3 + \frac{23}{45}x^5 - \frac{44}{105}x^7 + \frac{121}{315}x^9$&c.

4. It is required to find the square of the series $1+x+x^2+x^3+x^4$&c.

Ans. $1+2x+3x^2+4x^3+5x^4$&c.

5. It is required to find the square root of the series $1+x+x^2+x^3+x^4$&c.

Ans. $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4$&c.

6. It is required to find the square of the series $y-y^3+y^5-y^7+y^9$&c.

Ans. $y^2-2y^4+3y^6-4y^8+5y^{10}$&c.

7. It is required to find the 5th power of the series $2x^{\frac{1}{2}}+3x^{\frac{5}{2}}-4x^{\frac{7}{2}}+5x^{\frac{9}{2}}-6x^{\frac{11}{2}}$ &c.

Ans. $32x^{\frac{5}{2}}+240x^{\frac{9}{2}}-320x^{\frac{11}{2}}+1120x^{\frac{13}{2}}-$
$2400x^{\frac{15}{2}}$ &c.

8. It is required to find the square root of $\dfrac{1}{1-\frac{1}{2}x^2+\frac{1}{4}x^4-\frac{1}{6}x^6+\frac{1}{8}x^8\text{\&c.}}$

OF THE REVERSION OF SERIES.

(B) Reversion of series, is the finding the value of the root, or unknown quantity, contained in an infinite series, by means of another infinite series, in which some other quantity only is contained; the method of doing which is as follows:

Rule I.

If the series consist of all the powers of the unknown quantity, as $ax+bx^2+cx^3+dx^4+ex^5$&c. $=y$, substitute the particular values of the coefficients, in the given example, for *a*, *b*, *c*, *d*, &c. in the following formula, and the result will be the value of x required.

$$x = \frac{y}{a} - b\frac{y^2}{a^3} + (2b^2 - ac)\frac{y^3}{a^5} - (5b^3 - 5abc + a^2d)\frac{y^4}{a^7} + (14b^4 - 21ab^2c + 3a^2c^2 + 6a^2bd - a^3e)\frac{y^5}{a^9} - (42b^5 - 84ab^3c + 28a^2bc^2 + 28a^2b^2d - 7a^3cd - 7a^3be + a^4f)\frac{y^6}{a^{11}}$$

&c. (n)

Where it is to be observed, that, if y be a large number, the series will often diverge, and consequently, in that case, be of no practical use.

EXAMPLES.

1. Given $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}$ &c. $= z$, to find the value of x in terms of z.

(n) The formula for the conversion of series, here given, was one of Newton's first improvements in analysis, which he transmitted, in 1676, in a letter to Oldenburgh, at that time Secretary to the Royal Society, with directions to have it communicated to Leibnitz; which letter, together with another on the same subject, was afterwards printed in the *Commerium Epistolicum* of Collins, 1722.

Arbogast, in his work before quoted, p. 240, has given the reverted series in a form which renders the law of its continuation sufficiently obvious; but as the symbols which he employs are made to denote the operations that are to be performed in deriving the successive coefficients of the several terms from each other, instead of the coefficients themselves, his theorem cannot be exhibited in this place, for the reasons already mentioned.

It may be farther observed, that the reverted series has been computed and verified in numbers, as it appears, for the first time, by Rubbiani, an Italian analyst, as far as to $\frac{y^9}{a^{17}}$, or to the 9th term inclusively. See *Trigonométrie de Cagnoli*, p. 46, 2d edit. where the series is given to the extent abovementioned.

Here $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{3}$, $d=\frac{1}{4}$, $e=\frac{1}{5}$&c. and $y=z$.

Whence

$$\frac{1}{a}=1$$

$$-b=-\frac{1}{2}$$

$$+(2b^2-ac)=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

$$-(5b^3-5abc+a^2d)=-(\frac{5}{8}-\frac{5}{6}+\frac{1}{4})=-\frac{1}{24}$$

$$+(14b^4-21ab^2c+3a^2c^2+6a^2bd-a^3e)=\frac{1}{120}$$

$$-(42b^5-84ab^3c+28a^2bc^2+28a^2b^2d-7a^3cd-7a^3be+$$
$$a^4f)=-(\frac{21}{16}-\frac{7}{2}+\frac{14}{9}+\frac{7}{4}-\frac{7}{12}-\frac{7}{10}+\frac{1}{6})=-\frac{1}{720}$$

&c. &c.

Therefore

$$x=z-\frac{1}{2}z^2+\frac{1}{2.3}z^3-\frac{1}{2.3.4}z^4+\frac{1}{2.3.4.5}z^5-\frac{1}{2.3.4.5.6}z^6\text{&c.}$$

2. Given $x-x^2+x^3-x^4+x^5-x^6+x^7$&c.$=z$, to find the value of x in terms of z.

Ans. $x=z+z^2+z^3+z^4+z^5+z^6$&c.

3. Given $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\frac{x^7}{7}$&c.$=z$, to find the value of x in terms of z.

Ans. $x=z+\frac{1}{2}z^2+\frac{1}{2.3}z^3+\frac{1}{2.3.4}z^4+\frac{1}{2.3.4.5}z^5$&c.

4. Given $x-\frac{x^2}{2}+\frac{x^3}{4}-\frac{x^4}{8}+\frac{x^5}{16}$&c.$=y$, to find the value of x in terms of y.

Ans. $x=y+\frac{y^2}{2}+\frac{y^3}{4}+\frac{y^4}{8}+\frac{y^5}{16}+\frac{y^6}{32}$&c.

5. Given $y+\frac{y^2}{r}+\frac{2y^3}{3r^2}+\frac{2y^4}{3r^3}+\frac{2y^5}{3r^4}+\frac{32y^6}{45r^5}+\frac{244y^7}{315r^6}$&c.$=z$, to find the value of y in terms of z.

Ans. $y=z-\frac{z^2}{r}+\frac{4z^3}{3r^2}-\frac{7z^4}{3r^3}+\frac{14z^5}{3r^4}-\frac{452z^6}{45r^5}+\frac{7148z^7}{315r^6}$&c.

RULE II.

If the series consist of the odd powers of the unknown quantity only, as $ax+bx^3+cx^5+dx^7+ex^9$&c. $=y$, substitute the particular values of the coefficients, in any given example, for a, b, c, d, &c. in the following formula, and the result will be the value of x, as required.

$$x=\frac{y}{a}-b\frac{y^3}{a^4}+(3b^2-ac)\frac{y^5}{a^7}-(12b^3+a^2d-8abc)\frac{y^7}{a^{10}}+$$

$$(55b^4-55ab^2c+10a^2bd+5a^2c^2-a^3e)\frac{y^9}{a^{13}}\text{\&c.}$$

Where it is to be observed, that, in this case, as well as in the former, the given and reverted series must be both of the same form, or otherwise they are not convertible into each other.

EXAMPLES.

1. Given $x-\frac{1}{2.3}x^3+\frac{1}{2.3.4.5}x^5-\frac{1}{2.3.4.5.6.7}x^7+\frac{1}{2.3.4.5.6.7.8.9}x^9$&c. $=z$, to find the value of x in terms of z.

Here

$$a=1,\ b=-\frac{1}{6},\ c=\frac{1}{120},\ d=-\frac{1}{5040},\ e=\frac{1}{362880},\ y=z,$$

Whence

$$\frac{1}{a}=1$$

$$-b=\frac{1}{6}=\frac{1}{2.3}$$

$$+(3b^2-ac)=(\frac{1}{12}-\frac{1}{120})=\frac{3}{40}=\frac{3}{2.4.5}$$

$$-(12b^3+a^2d-8abc)=-(-\frac{1}{18}-\frac{1}{5040}+\frac{1}{90})=\frac{3.5}{2.4.6.7}$$

$$+(55b^4-55ab^2c+10a^2bd+5a^2c^2-a^3e)=(\frac{55}{1296}-$$

$$\frac{11}{864}+\frac{1}{3024}+\frac{1}{2880}-\frac{1}{362880})=\frac{3.5.7}{2.4.6.8.9}$$

Therefore

$$x = z + \frac{1}{2.3}z^3 + \frac{3}{2.4.5}z^5 + \frac{3.5}{2.4.6.7}z^7 + \frac{3.5.7}{2.4.6.8.9}z^9 \&c.$$

2. Given $x + x^3 + x^5 + x^7 + x^9 \&c. = y$, to find the value of x in terms of y.

Ans. $x = y - y^3 + 2y^5 - 5y^7 + 14y^9 \&c.$

3. Given $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} \&c. = z$, to find the value of x in terms of z.

Ans. $x = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{351}z^7 + \frac{62}{2835}z^9 \&c.$

4. Given $x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{61}{5040}x^7 + \frac{277}{72576}x^9 \&c. = z$, to find the value of x in terms of z.

Ans. $x = z - \frac{1}{6}z^3 + \frac{1}{24}z^5 - \frac{61}{5040}z^7 + \frac{277}{72576}z^9 \&c.$(o)

RULE III.

When two series are equal to each other, as $ax + bx^2 + cx^3 + dx^4 + ex^5 \&c. = \alpha y + \beta y^2 + \gamma y^3 + \delta y^4 + \varepsilon y^5 \&c.$ and it is required to find the value of the unknown quantity in one of them, in terms of the unknown quantity in the other, substitute the particular values of the coefficients in the given example, in the place of the known letters, in the following general formula, and the result will be the root, or value required.

(o) It is something remarkable, as has been observed by Maseres, *Script. Logarithmici*, Vol. III, p. 479, that the complicated series, in Example 4, which is the log. tan. $(45° + \frac{1}{2}x)$, should have its several terms the same, except with regard to the signs, as those of the reverted series; this being the only instance of such a series yet known.

$$x=\alpha\frac{y}{a}+(\beta-b\mathrm{A}^2)\frac{y^2}{a}+(\gamma-2b\mathrm{AB}-c\mathrm{A}^3)\frac{y^3}{a}+(\delta-b\mathrm{B}^2-2b\mathrm{AC}-3c\mathrm{A}^2\mathrm{B}-d\mathrm{A}^4)\frac{y^4}{a}+(\epsilon-2b\mathrm{BC}-2b\mathrm{AD}-3c\mathrm{AB}^2-3c\mathrm{A}^2\mathrm{C}-4d\mathrm{A}^3\mathrm{B}-e\mathrm{A}^5)\frac{y^5}{a}+(\zeta-2b\mathrm{BD}-b\mathrm{C}^2-2b\mathrm{AE}-c\mathrm{B}^3-6c\mathrm{ABC}-3c\mathrm{A}^2\mathrm{D}-6d\mathrm{A}^2\mathrm{B}^2-4d\mathrm{A}^3\mathrm{C}-5e\mathrm{A}^4\mathrm{B}-f\mathrm{A}^6)\frac{y^6}{a}\ \&c.$$

Where it is to be observed, that A, B, C, &c. are the coefficients of the first, second, third, &c. terms.

The same form may also be rendered equally applicable to the case in which the odd powers of x and y only are concerned, by putting each of the coefficients of the even powers $=0$.

EXAMPLES.

1. Given $x-\frac{1}{2}x^2+\frac{1}{2.3}x^3-\frac{1}{2.3.4}x^4+\frac{1}{2.3.4.5}x^5$ &c. $=\frac{1}{2}y+\frac{1}{3}y^2+\frac{1}{4}y^3+\frac{1}{5}y^4+\frac{1}{6}y^5$&c. to find x in terms of y.

Here $a=1$, $b=-\frac{1}{2}$, $c=\frac{1}{2.3}$, $d=-\frac{1}{2.3.4}$&c.; and $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $\gamma=\frac{1}{4}$, $\delta=\frac{1}{5}$, &c.

Whence

$\alpha=\frac{1}{2}=\mathrm{A}$

$\beta-b\mathrm{A}^2=\frac{1}{3}+\frac{1}{2}\times\frac{1}{4}=\frac{11}{24}=\mathrm{B}$

$\gamma-2b\mathrm{AB}-c\mathrm{A}^3=\frac{1}{4}+\frac{11}{48}-\frac{1}{48}=\frac{11}{24}=\mathrm{C}$

$\delta-b\mathrm{B}^2-2b\mathrm{AC}-3c\mathrm{A}^2\mathrm{B}-d\mathrm{A}^4=\frac{1381}{2880}=\mathrm{D}$

$\epsilon-2b\mathrm{BC}-2b\mathrm{AD}-3c\mathrm{AB}^2-3c\mathrm{A}^2\mathrm{C}-4d\mathrm{A}^3\mathrm{B}-e\mathrm{A}^5=\frac{1543}{3840}$

&c. &c.

Therefore $x=\frac{1}{2}y+\frac{11}{24}y^2+\frac{11}{24}y^3+\frac{1381}{2880}y^4+\frac{1543}{3840}$&c.

2. Given $x + x^2 + x^3 + x^4 + x^5$ &c. $= y + \frac{1}{2}y^2 + \frac{1}{4}y^3 + \frac{1}{8}y^4 + \frac{1}{16}y^5$ &c. to find x in terms of y.

Ans. $x = y - \frac{1}{2}y^2 + \frac{1}{4}y^3 - \frac{1}{8}y^4 + \frac{1}{16}y^5$ &c.

3. Given $x + \frac{x^3}{6d^2} + \frac{3x^5}{40d^4} + \frac{5x^7}{112d^6}$ &c. $= ny + \frac{ny^3}{6d^2} + \frac{3ny^5}{40d^4} + \frac{5ny^7}{112d^6}$ &c. to find x in terms of y.

Ans. $ny + \frac{1-n^2}{2.3d^2}\text{A}y^2 + \frac{9-n^2}{4.5d^2}\text{B}y^2 + \frac{25-n^2}{6.7}\text{C}y^2$ &c.

Where A, B, C, &c, are the foregoing terms with their proper signs.

OF THE METHOD OF INDETERMINATE COEFFICIENTS.

(c) THIS is a species of investigation, which is frequently used for obtaining the development of certain fractional and other expressions, without having recourse to the operations of division, or the extraction of roots; the method of performing which is as follows:

RULE.

Assume a series, or other expression, with unknown coefficients, for that which is required to be found; then, having multiplied it by the denominator of the given fraction, or raised it to its proper powers, find the value of each of these coefficients, by equating the homologous terms of the two expressions, or putting such of them as have no corresponding terms, equal to 0, as the case may require.

EXAMPLES.

1. Let it be required to find the development of $\frac{a}{a'+b'x}$, according to the above method.

$$\text{Assume } \frac{a}{a'+b'x} = A + Bx + Cx^2 + Dx^3 + Ex^4 \&c.$$

Then, multiplying the right hand side of the equation by $a'+b'x$, and transposing a, we shall have

$$0 = \begin{array}{r|r|r|l} Aa' + Ba' & x + Ca' & x^2 + Da' & x^3 \&c. \\ -a + Ab' & +Bb' & +Cb' & \end{array}$$

And by putting the first term, and the coefficients of the several powers of x, each $=0$, there will arise the following equations:

$$\begin{array}{r|c|l} Aa' - a = 0 & & A = \frac{a}{a'} \\ Ba' + Ab' = 0 & & B = -\frac{b'}{a'}A \\ Ca' + Bb' = 0 & \text{Or} & C = -\frac{b'}{a'}B \\ Da' + Cb' = 0 & & D = -\frac{b'}{a'}C \\ \&c. & & \&c. \end{array}$$

$$\text{Hence } \frac{a}{a'+b'x} = \frac{a}{a'} - \frac{b'}{a'}Ax - \frac{b'}{a'}Bx^2 - \frac{b'}{a'}Cx^3 \&c.$$

Where it is obvious, that each coefficient, in parting from the second inclusively, is equal to that which precedes it, multiplied by $-\frac{b'}{a'}$: which law renders it unnecessary to take a greater number of equations, or to push the calculation farther.

2. Required the development of $\frac{a+bx}{a'+b'x+c'x^2}$, according to the same method.

$$\text{Assume } \frac{a+bx}{a'+b'x+c'x^2} = A + Bx + Cx^2 + Dx^3 \&c.$$

Then multiplying the right hand side of the equation by $a'+b'x+c'x^2$, and transposing $a+bx$, we shall have

$$\begin{array}{r|r|r|l} 0 = \text{A}a' + \text{B}a' & x + \text{C}a' & x^2 + \text{D}a' & x^3 \&c. \\ -a + \text{A}b' & + \text{B}b' & + \text{C}b' & \\ -b & + \text{A}c' & + \text{B}c' & \end{array}$$

And by putting the coefficients of the several powers of $x=0$, there will arise the equations

$$\begin{array}{r|c|l} \text{A}a' - a = 0 & & \text{A} = \frac{a}{a'} \\ \text{B}a' + \text{A}b' - b = 0 & & \text{B} = -\frac{b'}{a'}\text{A} + \frac{b}{a'} \\ \text{C}a' + \text{B}b' + \text{A}c' = 0 & \text{Or} & \text{C} = -\frac{b'}{a'}\text{B} - \frac{c'}{a'}\text{A} \\ \text{D}a' + \text{C}b' + \text{B}c' = 0 & & \text{D} = -\frac{b'}{a'}\text{C} - \frac{c'}{a'}\text{B} \\ \&c. & & \&c. \end{array}$$

Whence $\frac{a+bx}{a'+b'x+c'x^2} =$

$$\frac{a}{a'} - \left(\frac{b'}{a'}\text{A} - \frac{b}{a'}\right)x - \left(\frac{b'}{a'}\text{B} + \frac{c'}{a'}\text{A}\right)x^2 - \left(\frac{b'}{a'}\text{C} + \frac{c'}{a'}\text{B}\right)x^3 \&c.$$

Where each coefficient, in parting from the third inclusively, may be readily deduced from the two that precede it. So that if P, Q, R, be any three consecutive coefficients, we shall have

$$\text{R}a' + \text{Q}b' + \text{P}c' = 0; \text{ or } \text{R} = -\frac{b'}{a'}\text{Q} - \frac{c'}{a'}\text{P}.$$

3. Given $(x^2+p)^2 - (qx+r)^2 = x^4 + ax^2 + bx + c$, to find the indefinite coefficients p, q, and r.

Here, by squaring the terms on the left hand side of the equation, and collecting those that are alike, we have

$$x^4 + (2p - q^2)x^2 - 2qrx + p^2 - r^2 = x^4 + ax^2 + bx + c.$$

And consequently, by equating the homologous terms,

$$\left.\begin{array}{r}2p-q^2=a\\ -2qr=b\\ p^2-r^2=c\end{array}\right| \quad \text{Or} \quad \left|\begin{array}{l}2p-a=q^2\\ \quad -b=2qr\\ p^2-c=r^2\end{array}\right.$$

Where it is plain, that the product of the first and third of these equations is equal to $\frac{1}{4}$ of the square of the second; or $2p^3-ap^2-2cp+ac=\frac{1}{4}b^2$.

Hence the value of p may be found by a cubic equation, and then q and r from the former equations.

4. It is required to convert $\frac{A}{b-ax}$ into a series by the above method.

Ans. $\frac{A}{b}(1+\frac{ax}{b}+\frac{a^2x^2}{b^2}+\frac{a^3x^3}{b^3}+\frac{a^4x^4}{b^4}+\&c.)$

5. It is required to convert $\frac{1+2x}{1-x-x^2}$ into a series by the above method.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+29x^6\&c.$

6. It is required to convert $\frac{1-x}{1-2x-3x^2}$ into a series by the same method.

Ans. $1+x+5x^2+13x^3+41x^4+121x^5+365x^6\&c.$

7. It is required to convert $\sqrt{(1-x)}$ into a series by the same method.

Ans. $1-\frac{x}{2}-\frac{x^2}{2.4}-\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}-\frac{3.5.7x^5}{2.4.6.8.10}\&c.$

8. It is required to find, according to the above method, the several roots, or values of x, in the equation $x^4-6x^3+13x^2-12x=a$, by means of a quadratic.

Ans. $x=\frac{3}{2}\pm\sqrt{\{\frac{1}{4}\pm\sqrt{(a+4)}\}}$

OF VANISHING FRACTIONS.

(D) Vanishing fractions, and other similar expressions, are such, as in certain cases, become equal to $\frac{0}{0}$; which symbol, though apparently nugatory, or of no value, must not be rejected as useless, being of frequent occurrence in several algebraical and fluxional investigations, where it will often be found, from the nature of the subject, to denote some fixed, or determinate quantity (*p*).

Thus, if *a* be made to represent the first term of any regular geometrical series, *r* the ratio, and *n* the number of terms, we shall have, by the rule in Art. M, p. 91,

$$\frac{ar^n - a}{r - 1} = a + ar + ar^2 + ar^3 + ar^4 + \ldots\ldots ar^{n-2} + ar^{n-1}$$

Where the left hand number of the equation is an universal expression for the sum (s) of the series, whatever may be the values of *a*, *r*, and *n*; as will appear by dividing the numerator by the denominator.

Let, therefore, the ratio, or multiplier *r*, be taken $= 1$; in which case, its power r^n, or 1^n, will, also, $= 1$; and the expression for the sum will be

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(*p*) The idea of fractions of this kind first originated about the year 1702, in a contest between Varignon and Rolle, two French mathematicians of considerable eminence, concerning the principles of the Differential Calculus, of which Rolle was a strenuous opposer. Among other arguments against the truth of this doctrine, which had then been recently introduced, he proposed an example of drawing a tangent to a certain curve, at the point where the two branches intersect each other; and as the fractional expression for the subtangent, according to that method, had both its numerator and denominator equal to 0, he regarded such a result as absurd, and adduced it as a proof of the falacy of this mode of solution. But the mystery was soon aft-

$$s = \frac{a-a}{1-1} = \frac{0}{0}$$

But when $r = 1$, the original series becomes of the form

$$s = a + a + a + a + \&c. \ldots \text{ to } n \text{ terms};$$

of which the sum is, evidently, $= na$; and, therefore, in this case, it follows, that $\frac{0}{0} = na$.

And in the same way it might be shown, that this symbol is the representative of various other quantities, according to the nature of the expression from which it is derived; but it will be here sufficient to observe, that the true value of any fractional expression of this kind may be readily obtained, as follows.

RULE.

1. If both the terms of the given fraction be rational, divide each of them by their greatest common measure; then, if the hypothesis which is found to reduce the original expression to the

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erwards explained by John Bernouilli, and, upon the renewal of the dispute, still farther by Saurin, who showed that $\frac{0}{0}$, in the case here mentioned, had a real value.—See *Montucla, Hist. Math.* Vol. III, p. 114.

These fractions were also the cause of a violent controversy between Waring and Powell, in 1760, when these gentlemen were candidates for the mathematical professorship at Cambridge; Waring maintaining that the fraction $\frac{x-x^5}{1-x}$, given as the second example of this article, is $=4$ when $x=1$, and Powell, (or rather Maseres, who is commonly thought to have conducted the dispute,) that it was $=0$: a full account of which may be seen in the pamphlets that passed between them at the time above mentioned.

form $\frac{0}{0}$, be applied to the result, it will give the true value of the fraction in the state under consideration.

2. When any part of the fraction is irrational, observe what the unknown quantity is equal to when the numerator and denominator both vanish, and put it = that quantity + or $-i$; then, if this be substituted for the unknown quantity, and the roots of the surds be extracted, to a sufficient number of places, the result, when i is put $=0$, will give the true value of the fraction.

EXAMPLES.

1. It is required to find the value of the fraction $\frac{x^2-a^2}{x-a}$, when x is equal to a.

Here, if we put $x=a$, there will arise $\frac{a^2-a^2}{a-a}=\frac{0}{0}$

But, by division, $\frac{x^2-a^2}{x-a}=x+a$; and if x be now put $=a$, we shall have $\frac{a^2-a^2}{a-a}=2a$; whence $\frac{0}{0}$, or the given fraction, in its vanishing state, is $=2a$.

2. It is required to find the value of the fraction $\frac{x-x^5}{1-x}$, when x is equal to 1.

Here, as before, if we put $x=1$, there will arise $\frac{1-1}{1-1}=\frac{0}{0}$. But, by division, $\frac{x-x^5}{1-x}=x+x^2+x^3+x^4$; where, if x be now put $=1$, we shall have $\frac{1-1}{1-1}=4$; which is, therefore, in this case, the true value of the symbol $\frac{0}{0}$.

3. It is required to find the value of the expression $y=\frac{b(x-\sqrt{ax})}{x-a}$, when x is equal to a.

Here, if x be taken $=a+i$, according to the rule, we shall have

$$y=\frac{b[a+i-\sqrt{(a^2+ai)}]}{i}$$

And, by extracting the square root of a^2+ai, and then dividing by i,

$$y=b\{\frac{1}{2}+\frac{1}{2.4}(\frac{i}{a})-\frac{3}{2.4.6}(\frac{i}{a})^2 \text{ \&c.}\}$$

Whence, putting the indeterminate quantity $i=0$, there will arise

$$y=\frac{1}{2}b,$$

Which is the true value of the expression in the case proposed.

4. It is required to find the value of the fraction $\frac{x^m-a^m}{x-a}$, when $x=a$.

$$\text{Here } \frac{x^m-a^m}{x-a}=$$

$$(x-a)^{m-1}+ma(x-a)^{m-2}+\frac{m(m-1)}{1\ .\ 2}a^2(x-a)^{m-3}+$$
$$\frac{m(m-1)(m-2)}{1\ .\ 2\ .\ 3}a^3(x-a)^{m-4}+\ \ldots\ldots\ +ma^{m-1}.$$

And if we take $x=a$, both in the original expression and in the quotient, there will arise

$$\frac{0}{0}=ma^{m-1};$$

where it is to be observed, that, by this hypothesis, all the terms of the quotient will vanish, except the last.

Whence the undeveloped fraction, which, in the supposed case, takes the form $\frac{0}{0}$, has, for its true value, the definite quantity ma^{m-1}.

5. Let there be taken, as another example of this kind, the following equation

$$y = \frac{P(x-a)^m}{Q(x-a)^n};$$

where P and Q are supposed to be certain functions, or combinations of x, which do not become 0 for the same value of x.

Then taking $x = a$, the expression, according to this hypothesis, will become of the form

$$\frac{P \times 0}{Q \times 0} = \frac{0}{0}.$$

But by considering the indices m, n, of the proposed fraction, under each of the relations

$$m > n, \ m = n, \ m < n,$$

we shall have, by division, the three following results:

$$y = \frac{P(x-a)^{m-n}}{Q}, \ y = \frac{P}{Q}, \ y = \frac{P}{Q(x-a)^{n-m}}.$$

And consequently, by now taking $x = a$, there will arise

$$y = \frac{P \times 0}{Q}, \ y = \frac{P}{Q}, \ y = \frac{P}{Q \times 0}.$$

Whence, agreeably to the subjoined note, the value of the symbol $\frac{0}{0}$, in this case, will be nothing, finite or infinite, according to the conditions above mentioned.

6. It is required to find the value of the fraction $\frac{x^3-a^3}{x-a}$, when x is equal to a. Ans. $3a^2$

7. It is required to find the value of the expression $\frac{1-x^n}{1-x}$, when x is equal to 1. Ans. n

8. It is required to find the value of the expression $\frac{a\sqrt{ax}-x^2}{a-\sqrt{ax}}$, when x is equal to a. Ans. $3a$

9. It is required to find the value of the expression $\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$, when x is equal to 1.

Ans. $\frac{n(n+1)}{2}$

10. It is required to find the value of the expression $\frac{\sqrt{x}-\sqrt{a}+\sqrt{(x-a)}}{\sqrt{(x^2-a^2)}}$, when x is equal to a.

Ans. $\frac{1}{\sqrt{2a}}$

11. It is required to find the value of the expression $\frac{(2a^3x-x^4)^{\frac{1}{2}}-a(a^2x)^{\frac{1}{3}}}{a-(ax^3)^{\frac{1}{4}}}$ when x is equal to a.

Ans. $\frac{16a}{9}$

Note. In addition to the above article, it will here be proper to show the signification of some other operations and symbols, which often occur in the solution of problems: as, for instance,

1. If 0 be multiplied, or divided, by any finite quantity, the product, or quotient, will be 0.

Thus $0 \times a = 0$, or $\frac{0}{a} = 0$.

2. If any finite quantity be divided by 0, the quotient will be infinite.

Thus $\frac{1}{0}$, or $\frac{a}{0} = \infty$.

3. Adding or subtracting any finite quantity, to, or from, an infinite quantity, makes no alteration in it.

Thus $\infty \pm a = \infty$.

OF FIGURATE AND POLYGONAL NUMBERS.

(E) *Figurate numbers,* are such as arise from taking the successive sums of the series of natural numbers 1, 2, 3, 4, &c.; and then the successive sums of these last, and so on: and polygonal numbers, are those which are formed of the successive sums of the terms of any arithmetical progression, beginning with unity; each of them being usually divided into orders, according to the scale of their generation; which, as far as regards those of the first class, may be shown as follows:

Order.	*Figurate Numbers.*	*General Terms.*
1	1, 2, 3, 4, 5, 6&c.	n
2	1, 3, 6, 10, 15, 21&c.	$\frac{n(n+1)}{1.2}$
3	1, 4, 10, 20, 35, 56&c.	$\frac{n(n+1)(n+2)}{1.2.3}$
4	1, 5, 15, 35, 70, 126&c.	$\frac{(n+1)(n+2)(n+3)}{1.2.3.4}$
&c.	&c.	&c.

Where it is to be observed that the general terms, here given, are so called, because if 1, 2, 3, &c. be respectively substituted, in each of them, for n, we shall obtain the several terms of the series.

And if, instead of the natural numbers 1, 2, 3, 4, &c. which give triangular numbers, an arithmetical series be taken, the common difference of which is 2, the sum of its successive terms will be the series of square numbers; if the common difference be 3, the series will be pentagonal numbers; if 4, hexagonal; and so on: thus,

Arith. Series.	*Ord.*	*Polygonal Numbers.*	*Gen. Terms.*
1, 2, 3, 4&c.	1	Tri. 1, 3, 6, 10&c.	$\frac{n(n+1)}{2}$
1, 3, 5, 7&c.	2	Sqrs. 1, 4, 9, 16&c.	$\frac{n(2n+0)}{2}$
1, 4, 7, 10&c.	3	Pent. 1, 5, 12, 22&c.	$\frac{n(3n-1)}{2}$
1, 5, 9, 13&c.	4	Hex. 1, 6, 15, 28&c.	$\frac{n(4n-2)}{2}$
&c.		&c.	&c.

Where the number denoting any order is the common difference of the arithmetical series, from which the polygonal numbers, belonging to that order, are generated.

In like manner, if we take the successive sums of the several polygonals thus obtained, and then the successive sums of these last, and so on, a great variety of other orders of series of this kind may be readily obtained (*q*).

Hence, also, in general, if n be made to denote the number of terms of the series, a figurate number of any order m, may be expressed by the following formula:

$$\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \; . \; . \; . \; . \; . \; \frac{n+(m-1)}{m}$$

And supposing n to be the number of terms of the series, as before, a polygonal number of the order $m-2$, or one that has the number of its sides denoted by m, may be expressed by

$$\frac{(m-2)n^2-(m-4)n}{2}$$

(*q*) The manner of representing polygonal numbers, by placing points, considered as their units, in the form of triangles, squares, &c. which first gave rise to the appellations they now bear, may be seen, by those who are curious in such matters, in Malcolm's Arithmetic, p. 397, and in Euler's Algebra, Vol. I, c. 5; but the truth is, that this fanciful mode of illustration has no necessary connexion with the subject in question; being one of the figments, still remaining, of the old Pythagorean doctrine respecting the mysterious nature of certain numbers and figures, which, together with many of the uncouth terms that are usually attached to it, ought to be wholly abolished. Besides which, it may be remarked, that, except those of the second order, the numbers usually called figurates, are not susceptible of this kind of representation; but arise solely from the expansion of a bino-

So that figurate numbers, of any order, may be always determined, without computing those of the preceding orders, by taking as many factors, in the first of these formulæ, as is denoted by m, or the number of the given order, and making n equal to the term that is to be found.

And a polygonal number of any order, or number of sides, may be readily ascertained from the second of these formulæ, by substituting the number denoting that order for $m-2$, or the number of sides of the polygon, for m, and taking n equal to the term required.

EXAMPLES.

1. Required the 15th term of the second order of figurate numbers, 1, 3, 6, 10, 15, &c.

Here m being $=2$, and $n=15$, we shall have, by the 1st formula,

$$\frac{n(n+1)}{2}=\frac{15(15+1)}{2}=\frac{15\times 16}{2}=15\times 8=120,$$

the term required.

2. It is required to find the 20th term of the 5th order of figurate numbers.

Here m being $=5$, and $n=20$, we shall have

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mial into a series, of the several terms of which they are the respective coefficients.

With respect to the theoretical part of this subject, it may be observed, that the nth term of any order of figurate numbers is equal to the $(n-1)$th term of that order, plus the nth term of the preceding order; which principle contains the whole doctrine of these numbers; and leads to an easy demonstration of a general proposition mentioned by Fermat, in his Notes on Diophantus, p. 16, which was considered by him as one of his principal discoveries.

$\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} = \frac{20}{1} \times \frac{21}{2} \times \frac{22}{3} \times \frac{23}{4} \times \frac{24}{5} =$ 42504, the term required.

3. It is required to find the 12th term of the fifth order of polygonal numbers, being those called heptagonal, or such as would be represented by a figure of seven sides.

Here m being $=7$, and $n=12$, we shall have, by the second formula,

$$\frac{(m-2)n^2-(m-4)n}{2} = \frac{(7-2)\times 144-(7-4)\times 12}{2} = 5\times 72 -$$

$3\times 6 = 360 - 18 = 342$, the term required.

4. It is required to find the 13th term of the 9th order of figurate numbers. Ans. 293930

5. It is required to find the 36th term of that order of polygonal numbers, which is denoted by a figure of twenty-five sides. Ans. 14526

6. It is required to find the first seven terms of the 6th order of figurate numbers.

7. It is required to find the first twelve terms of the order of polygonal numbers called nonagonal, or such as are denoted by a figure of nine sides.

OF THE INDETERMINATE ANALYSIS.

(F) In the common rules of algebra, such questions are usually proposed as require some certain or definite answer; in which case, it is necessary that there should be as many independent equations, expressing their conditions, as there are unknown quantities to be determined; or otherwise the problem would not be limited.

But in other branches of the science, questions frequently arise that involve a greater number of unknown quantities than there are equations to express them; in which case, they are called indeterminate or unlimited problems; being usually such as admit of a great variety of solutions; although when the answers are required only in whole positive numbers, they are generally confined within certain limits, and may be determined as follows:

PROBLEM I.

To find the values of the unknown quantities x and y in the equation

$$ax \pm by = c,$$

where a and b are given numbers which admit of no common divisor, except when it is, also, a divisor of c.

RULE.

1. Let *wh.* stand for a whole or integral number, and reduce the equation to the form $x = \frac{by + c}{a} = wh.$

2. Make $\frac{by + c}{a} = \frac{dy + e}{a}$, by throwing all whole numbers out of it, till d and e be each less than a.

3. Find the difference, or sum, of $\frac{dy + e}{a}$, or some multiple of it, and $\frac{ay}{a}$, or any other multiple of it that comes near the former, and the result will be a whole number.

4. Take this, or any multiple of it, from one of the foregoing fractions, or from any whole number which is nearly equal to it, and the result, in this case, will also be a whole number.

5. Proceed in the same manner with this last

result; and so on, till the coefficient of y becomes equal to 1, or $\frac{y+r}{a} = wh. = p$.

6. Then will $y = ap \pm r$; where p may be any whole number whatever, that makes y positive; and as the value of y is now known, that of x may be found from the given equation (r).

Note. Any indeterminate equation, of the form

$$ax + by = c,$$

is always possible when a, b, c, are prime to each other, and $c \asymp \{b(a-1) - a\}$; but the quantities x and y, in this case, can only have a determinate number of values.

Whereas, if the proposed equation be of the form

$$ax - by = \pm c,$$

it will admit of an infinite number of solutions (s).

(r) This rule is founded on the obvious principle, that the sum, difference, or product, of any two whole numbers, is a whole number; and that, if a number divides the whole of any other number, and a part of it, it will also divide the remaining part.

(s) That the coefficients a and b, when these two formulæ are possible, should have no common divisor, which is not, at the same time, a divisor of c, is evident; for if $a = md$, and $b = me$, we shall have $ax \pm by = mdx \pm mey = c$; and consequently $dx + ey = \frac{c}{m}$: but d, e, x, y, being supposed to be whole numbers, $\frac{c}{m}$ must also be a whole number, which it cannot be, except when m is a divisor of c.

Hence, if it were required to pay 100*l.* in guineas and moidores only, the question would be impossible; since, in the equation $21x + 27y = 2000$, which represents the conditions of the problem, the coefficients, 21 and 27, are each divisible by 3, whilst the absolute term 2000 is not divisible by it.

EXAMPLES.

1. Given $2x+3y=25$, to determine x and y in whole positive numbers.

$$\text{Here } x=\frac{25-3y}{2}=12-y+\frac{1-y}{2}$$

Hence, since x must be a whole number, it follows that $\frac{1-y}{2}$ must also be a whole number.

$$\text{Let therefore } \frac{1-y}{2}=wh.=p;$$

$$\text{Then } 1-y=2p, \text{ or } y=1-2p.$$

And since

$$x=12-y+\frac{1-y}{2}=12-(1-2p)+p=12+3p-1,$$

$$\text{We shall have } x=11+3p, \text{ and } y=1-2p;$$

Where p may be any whole number whatever, that will render the values of x and y in these two equations positive.

But it is evident, from the value of y, that p must be either 0 or negative; and, consequently, from that of x, that it must be -1, -2, or -3.

Whence, if $p=0$, $p=-1$, $p=-2$, $p=-3$,

$$\text{Then } \begin{cases} x=11, & x=8, & x=5, & x=2 \\ y=1, & y=3, & y=5, & y=7 \end{cases}$$

Which are all the answers in whole positive numbers that the question admits of.

2. Given $19x=14y-11$, to find x and y in whole numbers.

$$\text{Here } x=\frac{14y-11}{19}=wh., \text{ and also } \frac{19y}{19}=wh.$$

$$\text{Whence, by subtraction, } \frac{19y}{19}-\frac{14y-11}{19}=\frac{5y+11}{19}=wh.$$

$$\text{Also, } \frac{5y+11}{19}\times 4=\frac{20y+44}{19}=y+2+\frac{y+6}{19}=wh.$$

And by rejecting $y+2$, which is a whole number,

$$\frac{y+6}{19}=wh.=p.$$

Therefore $y=19p-6$,

$$\text{And } x=\frac{14y-11}{19}=\frac{14(19p-6)-11}{19}=\frac{266p-95}{19}=14p-5.$$

Whence, if p be taken $=1$, we shall have $x=9$ and $y=13$, for their least values; the number of solutions being obviously indefinite.

3. Given $3x=8y-16$, to find the values of x and y in whole numbers.

Here $x=\frac{8y-16}{3}=2y-5+\frac{2y-1}{3}=wh.$; or $\frac{2y-1}{3}=wh.$

$$\text{Also } \frac{2y-1}{3}\times 2=\frac{4y-2}{3}=y+\frac{y-2}{3}=wh.$$

And, by rejecting y, which is a whole number, there will remain $\frac{y-2}{3}=wh.=p.$

Therefore $y=3p+2$,

$$\text{And } x=\frac{8y-16}{3}=\frac{8(3p+2)-16}{3}=\frac{24p}{3}=8p.$$

And, if p be put $=1$, we shall have $x=8$ and $y=5$, for their least values, the number of answers, as in the former question, being evidently indefinite.

4. Given $21x+17y=2000$, to find all the possible values of x and y in whole numbers.

$$\text{Here } x=\frac{2000-17y}{21}=95+\frac{5-17y}{21}=wh.;$$

Or, omitting the 95, $\frac{5-17y}{21}=wh$;

Consequently, by addition, $\frac{21y}{21}+\frac{5-17y}{21}=\frac{4y+5}{21}=wh.$

$$\text{Also, } \frac{4y+5}{21}\times 5=\frac{20y+25}{21}=1+\frac{4+20y}{21}=wh.;$$

Or, by rejecting the whole number 1, $\frac{4+20y}{21}=wh.$

$$\text{And, by subtraction, } \frac{21y}{21} - \frac{4+20y}{21} = \frac{y-4}{21} = wh. = p;$$

$$\text{Whence } y = 21p + 4,$$

$$\text{And } x = \frac{2000-17y}{21} = \frac{2000-17(21p+4)}{21} = 92 - 17p.$$

Where, if p be put $=0$, we shall have the least value of $y=4$, and the corresponding, or greatest value of $x=92$.

And the rest of the answers will be found by adding 21 continually to the least value of y, and subtracting 17 from the greatest value of x; which being done we shall obtain the six following results:

$x=92$	75	58	41	24	7
$y=4$	25	46	67	88	109

These being all the solutions the question admits of.

Note 1. When there are three or more unknown quantities, and only one equation by which they can be determined, it will be proper first to find the limit of that quantity which has the greatest coefficient, and then to ascertain the different values of the rest, by separate substitutions of the several values of the former, from 1 up to the extent required, as in the following question.

5. Given $3x+5y+7z=100$, to find all the different values of x, y, and z, in whole numbers(*t*).

(*t*) When the proposed equation is of the form $ax+by+cz=d$, it may be observed, that if any of the coefficients, a, b, c, be negative, the equation may be put under the form

$$ax+by=d+cz,$$

from which, by means of the indeterminate quantity z, an indefinite number of values may be given to the second side of the equation.

Here each of the least integer values of x and y are 1, by the question; whence it follows, that

$$z=\frac{100-5-3}{7}=\frac{100-8}{7}=\frac{92}{7}=13\tfrac{1}{7}.$$

Consequently z cannot be greater than 13, which is also the limit of the number of answers; though they may be considerably less.

By proceeding, therefore, as in the former rule, we shall have

$$x=\frac{100-5y-7z}{3}=33-y-2z+\frac{1-2y-z}{3}=wh.;$$

And, by rejecting $33-y-2z$,

$$\frac{1-2y-z}{3}=wh.; \text{ or } \frac{3y}{3}+\frac{1-2y-z}{3}=\frac{y+1-z}{3}=wh.=p.$$

Whence $y=3p+z-1$; and, putting $p=0$, we shall have the least value of $y=z-1$; where z may be any number, from 1 up to 13, that will answer the conditions of the question.

When, therefore, $z=1$ we have $y=0$,

$$\text{And } x=\frac{100-7}{3}=31.$$

And by taking $z=2$, 3, 4, 5, &c. the corresponding values of x and y, together with those of z, will be found to be as below.

$z=$	1	2	3	4	5	6	7	8
$y=$	0	1	2	3	4	5	6	7
$x=$	31	27	23	19	15	11	7	3

Which are all the integer values of x, y, and z, that can be obtained from the given equation.

Note 2. If there be three unknown quantities,

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and consequently, also, to x and y in the first. And if in any equation of this form, a, b, c, have a common divisor, while d has not, the question, as in the first case, becomes impossible.

and only two equations, exterminate one of these quantities in the usual way, and find the values of the other two from the resulting equation, as before; then, if the values, thus found, be separately substituted, in either of the given equations, the corresponding values of the remaining quantities will likewise be determined.

6. Given $x-2y+z=5$, and $2x+y-z=7$, to find the values of x, y, and z.

Here, by multiplying the first of these equations by 2, and subtracting the second from the product, we shall have

$$3z-5y=3, \text{ or } z=\frac{3+5y}{3}=1+y+\frac{2y}{3}=wh.;$$

$$\text{And consequently } \frac{2y}{3}, \text{ or } \frac{3y}{3}-\frac{2y}{3}=\frac{y}{3}=wh.=p,$$

$$\text{Whence } y=3p.$$

And, by taking $p=0$, 1, 2, 3, 4, &c. we shall have $y=0$, 3, 6, 9, 12, 15, &c. and $z=1$, 6, 11, 16, 21, 26, &c.

But from the first of the two given equations

$$x=5+2y-z;$$

whence, by substituting the above values for y and z, the results will give $x=4$, 5, 6, 7, 8, 9, &c.

And therefore the first six values of x, y, and z, are as below:

$x=4$	5	6	7	8	9
$y=0$	3	6	9	12	15
$z=1$	6	11	16	21	26

Where the law by which they can be continued is sufficiently obvious.

7. Given $3x=8y-16$, to find the least values of x and y in whole numbers. Ans. $x=8$, $y=5$

8. Given $14x = 5y + 7$, to find the least values of x and y in whole numbers. Ans. $x = 3$, $y = 7$

9. Given $27x = 1600 - 16y$, to find the least values of x and y in whole numbers. Ans. $x = 48$, $y = 19$

10. It is required to divide 100 into two such parts, that one of them may be divisible by 7, and the other by 11.

Ans. The only parts are 56 and 44

11. Given $11x + 5y = 254$, to find all the possible values of x and y in whole numbers.

Ans. $\begin{cases} x = 19, \ 14, \ 9, \ 4 \\ y = 9, \ 20, \ 31, \ 42 \end{cases}$

12. Given $9x + 13y = 2000$, to find the least value of x and the greatest value of y in whole numbers. Ans. $x = 7$, $y = 149$

13. Given $17x + 19y + 21z = 400$, to find all the answers in whole numbers which the question admits of. Ans. 10 different answers

14. Given $5x + 7y + 11z = 224$, to find all the possible values of x, y, and z, in whole positive numbers. Ans. The number of answers is 59

15. It is required to find in how many different ways is it possible to pay 20*l.* in half-guineas and half-crowns, without using any other sort of coin?

Ans. 7 different ways

16. A person bought as many ducks and geese, together, as cost him 28*s.*; for the geese he paid 4*s.* 4*d.* a piece, and for the ducks 2*s.* 6*d.* a piece: what number had he of each?

Ans. 3 geese and 6 ducks

17. A person, in exchange for a number of pieces of foreign gold, valued at 17*s.* 4*d.* each, received

a certain number of guineas under 50, and 1*s.* over; what was the sum exchanged? Ans. 49*l.* 8*s.*

18. I owe my friend a shilling, and have nothing about me but guineas, and he has nothing but louisd'ors; how must I contrive to acquit myself of the debt, the louis being valued at 17*s.* a piece, and the guineas at 21*s.*?

Ans. I must give him 13 guineas, and he must give me 16 louis

19. Forty-one persons, consisting of men, women, and children, spent, among them, 40*s.*, of which each man paid 4*s.*, each woman 3*s.*, and each child 4*d.*; how many were there of each?

Ans. 5 men, 3 women, and 33 children

20. It is required to discharge a debt of 351*l.* with guineas and moidores only, so that there may be the least number of pieces of each sort; and to find what the whole will amount to, when paid every way the question admits of?

Ans. N° of ways 36, and whole amount 12636*l.*

21. How many gallons of British spirits, at 12*s.*, 15*s.*, and 18*s.* a gallon, must a rectifier of compounds take to make a mixture of 1000 gallons, that shall be worth 17*s.* a gallon?

Ans. $111\frac{1}{9}$ at 12*s.*, $111\frac{1}{9}$ at 15*s.*, and $777\frac{7}{9}$ at 18*s.*

22. In how many different ways can a refiner mix three kinds of silver, of $11\frac{1}{4}$, $13\frac{3}{4}$, and $17\frac{1}{2}$ pennyweights fine per ounce, so as to form a mass of 75 ounces of 15 pennyweights fine per ounce?

Ans. 8 different ways

23. A person bought 100 head of cattle for 200*l.*; *viz.* oxen at 20*l.* a piece, cows at 10*l.* a piece, calves

at 4*l.* a piece, and sheep at 1*l.* a piece; how many of each sort did he purchase?

Ans. The question admits of 10 different answers, one of which is 4 oxen, 1 cow, 5 calves, and 90 sheep

24. It is required to determine in how many different ways it is possible to pay 1000*l.* in crowns, guineas, and moidores only?

Ans. 70734 different ways

PROBLEM II.

To find such a whole number, as, being divided by other given numbers, shall leave given remainders.

Rule.

1. Call the number to be determined x, the numbers by which it is to be divided a, b, c, &c. and the given remainders f, g, h, &c.

2. Subtract each of the remainders from x, and divide the differences by a; and there will arise $\frac{x-f}{a}$, $\frac{x-g}{a}$, $\frac{x-h}{a}$, &c. = whole numbers.

3. Put the first of these fractions $\frac{x-f}{a}=p$, and substitute the value of x, as found from this equation, in the place of x in the second fraction.

4. Find the least value of p in this second fraction, by the last problem, which put $=r$, and substitute the value of x, in terms of r, in the place of x in the third fraction.

Find, in like manner, the least value of r, in this third fraction, which put $=s$, and substitute the value of x, in terms of s, in the fourth fraction, as before; and so on, to the last; when the value of x, thus found, will give the whole number required.

EXAMPLES.

1. It is required to find the least whole number, which, being divided by 17, shall leave a remainder of 7, and when divided by 26, shall leave a remainder of 13.

Let $x =$ the number required.

Then $\frac{x-7}{17}$ and $\frac{x-13}{26} =$ whole numbers.

And, putting $\frac{x-7}{17} = p$, we shall have $x = 17p + 7$,

Which value of x, being substituted in the second fraction, gives $\frac{17p+7-13}{26} = \frac{17p-6}{26} = wh.$

But it is obvious that $\frac{26p}{26}$ is also $= wh.$

And consequently $\frac{26p}{26} - \frac{17p-6}{26} = \frac{9p+6}{26} = wh.$

Or $\frac{9p+6}{26} \times 3 = \frac{27p+18}{26} = p + \frac{p+18}{26} = wh.$

And, by rejecting p, there remains $\frac{p+18}{26} = wh. = r.$

Therefore $p = 26r - 18$;

Where, if r be taken $= 1$, we shall have $p = 8$.

And consequently $x = 17p + 7 = 17 \times 8 + 7 = 143$, the number required.

2. It is required to find the least whole number, which, being divided by 11, 19, and 29, shall leave the remainders 3, 5, and 10.

Let $x =$ the number required.

Then $\frac{x-3}{11}$, $\frac{x-5}{19}$, and $\frac{x-10}{29}=$ whole numbers.

And, putting $\frac{x-3}{11}=p$, we shall have $x=11p+3$.

Which value of x, being substituted in the second fraction, gives $\frac{11p-2}{19}=wh.$

$$\text{Or } \frac{11p-2}{19}\times 2=\frac{22p-4}{19}=p+\frac{3p-4}{19}=wh.$$

And, by rejecting p, there will remain $\frac{3p-4}{19}=wh.$

Also by mult$^{\text{n}}$ $\frac{3p-4}{19}\times 6=\frac{18p-24}{19}=\frac{18p-5}{19}-1=wh.$

Or, by rejecting the 1, $\frac{18p-5}{19}=wh.$

But $\frac{19p}{19}$ is likewise $=wh.$

Whence $\frac{19p}{19}-\frac{18p-5}{19}=\frac{p+5}{19}=wh$, which put $=r$.

Then we shall have

$p=19r-5$, and $x=11(19r-5)+3=209r-52$.

And, by substituting this value of x in the third fraction, there will arise

$$\frac{209r-52}{29}=7r-2+\frac{6r-4}{29}=wh.$$

Or, by neglecting $7r-2$, we shall have the remaining part of the expression $\frac{6r-4}{29}=wh.$

But, by multiplication,

$$\frac{6r-4}{29}\times 5=\frac{30r-20}{29}=r+\frac{r-20}{29}=wh.$$

Or, by rejecting r, there will remain $\frac{r-20}{29}=wh.$ which put $=s$.

Then $r=29s+20$; and, by taking $s=0$, we shall have $r=20$.

And consequently

$$x = 209r - 52 = 209 \times 20 - 52 = 4128,$$

the number required.

3. To find a number, which, being divided by 6, shall leave the remainder 2, and when divided by 13, shall leave the remainder 3. Ans. 68

4. It is required to find the least whole number, which, being divided by 39, shall leave the remainder 16, and when divided by 56, the remainder shall be 27. Ans. 1147

5. It is required to find the least whole number, which, being divided by 7, 8, and 9, respectively, shall leave the remainders 5, 7, and 8. Ans. 1727

6. It is required to find the least whole number, which, being divided by 3, 5, 7, and 2, shall leave the remainders 2, 4, 6, and 0, respectively.

Ans. 104

7. To find three numbers in the proportion of 5, 7, and 9, which, being severally divided by 11, 13, and 15, shall leave the remainders 1, 2, 3, respectively.

Ans. The three least No^s are 2685, 3759, & 4833

8. It is required to find the least whole number, which, being divided by each of the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, shall leave no remainders.

Ans. 2520.

9. A person receiving a box of oranges, observed, that, when he told them out by 2, 3, 4, 5, and 6 at a time, he had none remaining; but when he told them out by 7 at a time, there remained 5: how many oranges were there in the box?

Ans. 180

10. A person having, by accident, broken a hamper of eggs, offered to pay for them on the spot, if the owner could tell how many he had; to which he replied, that he only knew there were between 300 and 400, and that, when he counted them by 2's, 3's, and 5's at a time, there always remained 1; but when he counted them by 7's at a time, there remained nothing: what number of eggs had he? Ans. 301

11. It is required to determine in what year of Christ the cycle of the sun was 8, the cycle of the moon 10, and the cycle of indiction 10; or, which is the same thing, to find the least whole number, that, being severally divided by 28, 19, and 15, shall leave the remainders 8, 10, and 10; the whole of these periods or cycles being 28, 19, and 15 years respectively. Ans. In the year 1567

PROBLEM III.

Of compound indeterminate equations.

Equations of this kind, not higher than the second degree, which admit of answers in whole numbers, are chiefly such as consist of the products, or squares, of two unknown quantities, together with the quantities themselves; being usually, as far as regards the plan of the present performance, of one of the four general forms given in the following rule.

RULE.

1. If the equation be of the form $xy = ax + by + c$, we shall have, for its solution in whole numbers, $y = a + \frac{ab + c}{x - b}$; where $x - b$ must be a divisor of $ab + c$.

2. If the equation be of the form $x^2 + xy = ax + by + c$, we shall have $y = -x + a - b + \frac{c + b(a-b)}{x-b}$; where $x - b$ must be a divisor of $c + b(a - b)$.

3. If the equation be $x^2 = y^2 + ay + b$, we shall have $y = \frac{a^2 - 4b}{8n} + \frac{n-a}{2}$, and $x = \frac{a}{2} + y - n$; where a and n must be even numbers, and n be so taken that $8n$ may be a divisor of $a^2 - 4b$.

4. And if the equation be $x^2 = ay^2 + by + c^2$, we shall have $y = \frac{b - 2cn}{n^2 - a}$, and $x = c + ny$; where n must be some whole number between $\sqrt{a}$ and $\frac{b}{2c}$ (u).

EXAMPLES.

1. Given $xy = 42 - 2x - 3y$, to find the several values of x and y in whole numbers.

Here, by the first form, $a = -2$, $b = -3$, & $c = 42$.

$$\text{Whence } y = -2 + \frac{6 + 42}{x + 3} = -2 + \frac{48}{x + 3}.$$

Where it is plain, that x must be such a number, that, when added to 3, it shall be a divisor of 48. But the divisors of 48, that will give quotients greater than 2, are 16, 12, 8, 6, 4, and 2.

(u) A variety of other formulæ, besides those given above, are equally susceptible of rational results, either in whole numbers or in fractions; but in general the artifices and train of reasoning that are required to be used for this purpose, depend upon principles which could not be conveniently explained in a work like the present. We must therefore refer those who are desirous of farther information on this curious and interesting subject to the notes of Lagrange, at the end of Euler's Algebra, or to the *Essai sur la Théorie des Nombres*, by Legendre, 2d edit. where they will find almost every branch of the Indeterminate Analysis treated with great perspicuity and elegance. See, also, for this purpose, Art. H of the present performance.

And consequently the integral values of the two unknown quantities are

$$x = 16 - 3, \text{ or } 13 \mid = 12 - 3, \text{ or } 9 \mid = 8 - 3, \text{ or } 5 \mid$$
$$= 6 - 3, \text{ or } 3 \mid = 4 - 3, \text{ or } 1$$

$$y = \frac{48}{16} - 2, \text{ or } 1 \mid = \frac{48}{12} - 2, \text{ or } 2 \mid = \frac{48}{8} - 2, \text{ or } 4 \mid$$
$$= \frac{48}{6} - 2, \text{ or } 6 \mid = \frac{48}{4} - 2, \text{ or } 10$$

Which are all the answers in whole positive numbers that the question admits of.

2. Given $x^2 + xy = 2x + 3y + 29$, to find the values of x and y in whole positive numbers.

Here, by the second form, $a = 2$, $b = 3$, & $c = 29$.

$$\text{Whence } y = -x - 1 + \frac{29 + 3(2-3)}{x - b} = -x - 1 + \frac{26}{x - 3}$$

Where it is plain, that x must be such a number, that, when diminished by 3, shall be a divisor of 26. But the several divisors of 26 are 1, 2, 13, and 26, the only ones of which, that will render the expression positive, are 1 and 2.

$$\text{Therefore } x = 1 + 3 = 4 \mid = 2 + 3 = 5,$$

$$\text{And } y = -4 - 1 + \frac{26}{4 - 3} = 21 \mid = -5 - 1 + \frac{26}{5 - 3} = 7.$$

Which are all the answers, in whole numbers, that the question admits of.

3. Given $x^2 = y^2 + 20y$, to find the values of x and y in whole positive numbers.

Here, by the third form, $a = 20$, and $b = 0$,

Whence

$$y = \frac{400}{8n} + \frac{n - 20}{2} = \frac{50}{n} + \frac{n}{2} - 10, \text{ and } x = 10 + y - n.$$

Where it is plain, that n must be some even number which is a divisor of 50.

But the only number of this kind, that will give positive results, is 2.

Therefore

$$y=\frac{50}{2}+1-10=16, \text{ and } x=10+16-2=24.$$

4. Given $x^2=5y^2-12y+64$, to find the values of x and y in whole positive numbers.

Here, by the 4th form, $a=5$, $b=-12$, & $c=8$.

Whence $y=\frac{-12-16n}{n^2-5}=\frac{16(n-\frac{3}{4})}{5-n^2}$ and $x=8+ny$.

Where it is plain, that n must be less than $\sqrt{5}$, and greater than $\frac{3}{4}$; which numbers are only 1 and 2.

$$\text{Therefore } y=\frac{-12-16}{1-5}=7 \;\Big|\; =\frac{-12-32}{4-5}=44,$$

$$\text{And } x=8+1\times 7=15 \;\Big|\; =8+2\times 44=96.$$

5. It is required to find two numbers, such, that their product, added to their sum, shall be 79.

$$\text{Ans.} \left\{\begin{array}{llll} 1, & 3, & 4, & 7 \\ 39, & 19, & 15, & 19 \end{array}\right.$$

6. Given $x^2+xy=4x+3y+27$, to find the several values of x and y in whole numbers.

$$\text{Ans.} \left\{\begin{array}{l} x=\ 4,\ \ 5, \text{ and } 6 \\ y=27,\ 11, \text{ and } 5 \end{array}\right.$$

7. Given $x^2=y^2+100y+1000$, to find the two least values of x and y in whole numbers.

Ans. $x=70$ and $y=30$

8. Given $x^2=50y^2+100y+100$, to find the values of x and y in whole numbers.

Ans. $x=190$ and $y=40$

OF THE DIOPHANTINE ANALYSIS.

(G) This branch of Algebra, which is so called from its inventor, Diophantus, a Greek mathematician of Alexandria in Egypt, who flourished in or about the third century after Christ, relates chiefly to the finding of square, cube, and other similar numbers, or to the rendering certain compound expressions free from surds; the method of doing which is by making such substitutions for the unknown quantity, as will reduce the resulting equation to a simple one, and then finding the value of that quantity in terms of the rest.

It is to be observed, however, that questions of this kind do not always admit of answers in rational numbers, and that, when they are resolvable in this way, no rule can be given that will apply in all the cases that may occur; but, as far as respects a particular class of these problems, relating to squares, they may generally be determined by means of some of the rules derived from the following formula; observing, that the possibility or impossibility of the different cases of it depend upon the nature of the coefficients a, b, c. (x)

PROBLEM I.

To render surd quantities of the form $\surd(a+bx+cx^2)$ rational; or, to find such values of x as will make $a+bx+cx^2$ a square.

(x) The subject here treated of, which forms one of the most curious and abstruse branches of the Indeterminate Analysis, has been amply investigated and exemplified, both by Diophantus and several modern writers on Algebra, whose works abound with inquiries of this nature; but as some of the most valuable of these performances have now become extremely scarce, and others are too voluminous to be brought into general use, it is hoped that the following methodical abstract of this part of the science, which comprehends most of the methods hitherto known, for resolving problems of this kind, will be found acceptable to such readers, as may wish, by means of a ready compendium, to acquire some knowledge of this interesting branch of the Analytic Art.

RULE.

1. In the case where $c=0$, put the remaining part $\sqrt{(a+bx)}=n$, or $a+bx=n^2$; and we shall have $x=\frac{n^2-a}{b}$; where n, both in this and the following cases, may be any number, either integral or fractional, that will render the value of x positive.

2. In the case where $a=0$, put $\sqrt{(bx+cx^2)}=nx$, or $bx+cx^2=n^2x^2$, then, by dividing by x, and transposing the terms, we shall have $n^2x-cx=b$; and consequently $x=\frac{b}{n^2-c}$.

3. When a is a square number, put it $=d^2$, and make $\sqrt{(d^2+bx+cx^2)}=d+nx$; then $d^2+bx+cx^2=d^2+2dnx+n^2x^2$, or $b+cx=2dn+n^2x$; and consequently $x=\frac{2dn-b}{c-n^2}$. Or, if $b=0$, $x=\frac{2dn}{c-n^2}$.

4. When c is a square number, put it $=e^2$, and make $\sqrt{(a+bx+e^2x^2)}=n+ex$; then $a+bx+e^2x^2=n^2+2enx+e^2x^2$, or $a+bx=n^2+2enx$; and consequently $x=\frac{a-n^2}{2en-b}$. Or, if $b=0$, $x=\frac{a-n^2}{2en}$.

5. When neither a nor c are square numbers, yet if the formula can be resolved into two simple factors, (which it always can when b^2-4c is a square, but not otherwise,) the irrationality of it may be taken away, by putting $\sqrt{(a+bx+cx^2)}=\sqrt{\{(d+ex)(f+gx)\}}=n(d+ex)$; in which case we shall have $(d+ex)(f+gx)=n^2(d+ex)^2$, or $f+gx=n^2(d+ex)$; and consequently $x=\frac{dn^2-f}{g-en^2}$. Or, if $d=0$, $x=\frac{f}{en^2-g}$, and if $f=0$, $x=\frac{dn^2}{g-en^2}$.

6. When neither of the above rules will apply, if

the formula can be resolved into two parts, one of which is a square, and the other the product of any two simple factors, the irrationality of it may be destroyed, by putting $\sqrt{(a+bx+cx^2)} = \sqrt{\{(d+ex)^2 + (f+gx)(h+kx)\}} = (d+ex) + n(f+gx)$; in which case we shall have $(d+ex)^2 + (f+gx)(h+kx) = (d+ex)^2 + 2n(d+ex)(f+gx) + n^2(f+gx)^2$, or $h + kx = 2n(d+ex) + n^2(f+gx)$; and consequently $x = \frac{n(2d+fn)-h}{k-n(2e+gn)}$. Or, if $d=0$, $x = \frac{fn^2-h}{k-n(2e+gn)}$.

Or, if the part, in this case, which is found to be a square, be a known quantity, put $\sqrt{(a+bx+cx^2)} = \sqrt{\{d^2 + (e+fx)(g+hx)\}} = d + n(e+fx)$; then we shall have $d^2 + (e+fx)(g+hx) = d^2 + 2dn(e+fx) + (e+fx)^2$, or $g + hx = 2dn + (e+fx)$; and consequently, by transposing and uniting the different terms, $x = \frac{e+2dn-g}{h-f}$.

7. These being all the cases of the general formula that are resolvable by any direct rule, it only remains to observe, that, in other instances of a different kind, if we can, by trials, find any one simple value of the unknown quantity that satisfies the conditions of the question, an expression may be derived from this that will furnish as many other values of it as we please.

Thus, let p be a value of x so found, and make $a + bp + cp^2 = q^2$; then, by putting $x = p + y$, we shall have $a + bx + cx^2 = a + b(p+y) + c(p+y)^2 = a + bp + cp^2 + by + 2cpy + cy^2$, or $a + bx + cx^2 = q^2 + (b+2cp)y + cy^2$; from which latter form the value of y, and consequently that of x, may be found as in Case 3.

Or, because $a=q^2-bp-cp^2$, if this value be substituted for a in the original formula $a+bx+cx^2$, it will become $q^2+b(x-p)+c(x^2-p^2)=q^2+(x-p)\times(b+cp+cx)$, which expression can be resolved by Case 6.

Note. By putting the general formula $\surd(a+bx+cx^2)=y$, we shall have $2cx+b=\surd\{4cy+(b^2-4ac)\}$; where it is obvious, that, if this latter expression can be rendered rational, the former will also be rational; and as $\surd\{4cy^2+(b^2-4ac)\}$, or the equivalent form $\surd(dy^2+e)$, consists, in this case, only of two terms, the possibility or impossibility of resolving the question, in this simple state of it, may be more easily perceived (y).

EXAMPLES.

1. It is required to find a number such, that if it be multiplied by 5, and then added to 19, the result shall be a square.

(y) Among an infinite number of formulæ of the kind here treated of, that are irresolvable, we may reckon $3x^2+2$, $3x^2-1$, $5x^2+3$, $11x^2+7$, &c. which can never become squares, whatever number, either whole or fractional, be substituted for x; though there are a great variety of cases, in which x may be assigned, even in integer numbers, so as to render the form ax^2+b a square. But the method of showing in what way this can be done, as well as the finding of integral answers in general, requires researches of a very intricate nature, which do not admit of being discussed in a work like the present. The reader, therefore, who wishes to pursue this inquiry, is referred to the second volume of Euler's Algebra, where that part of the subject is treated of at considerable length.

It may here, however, be proper to observe, that any expression of the form $ax^2+by^2=z^2$, in which a and b have no common

Let $x=$ the required number; then, as in Case I,

$$5x+19=n^2,\text{ or }x=\frac{n^2-19}{5};$$

where n may be any number whatever greater than $\sqrt{19}$.

Whence, if n be taken $=5, 6, 7$, respectively, we shall have

$$x=\frac{25-19}{5}=1\tfrac{1}{5},\text{ or }\frac{36-19}{5}=3\tfrac{2}{5},\text{ or }\frac{49-19}{5}=6;$$

the latter of which is the least value of x, in whole numbers, that will answer the conditions of the question; and consequently

$$5x+19=5\times6+19=30+19=49,$$

a square number, as was required.

2. It is required to find an integral number such, that it shall be both a triangular number and a square.

It is here first to be observed, that all triangular numbers are of the form $\frac{x^2+x}{2}$; and therefore the question is reduced to the making $\frac{x^2+x}{2}$, or its equal

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divisor, and neither of them contains any square factor, is always resolvable, if we can find two whole numbers n, n', such that $\frac{n^2-b}{a}$ and $\frac{n'^2-a}{b}$, shall, also, be whole numbers; and that, if these conditions cannot be fulfilled, the question is impossible. See *Essai sur la Théorie des Nombres*, by Legendre, 2d edit. p. 41.

If the coefficients a, b, in the general expression above given, should have square factors, as in the form $a'm^2x^2+b'n^2y^2=z^2$, let $mx=x'$, and $ny=y'$, and there will arise, by substitution, $a'x'^2+b'y'^2=z^2$; which being now resolved according to the rule, we shall have, for the original equation, $x=\frac{x'}{m}$ and $y=\frac{y'}{n}$; so that, when thus prepared, this case is equally possible with the former, provided a' and b' be prime to each other.

$\frac{2x^2+2x}{4}$, a square. But since a square number, when multiplied, or divided, by a square number, is still a square, it is the same thing as if it were required to make $2x^2+2x$ a square.

Let therefore $2x^2+2x=\frac{m^2x^2}{n^2}$, agreeably to Case 2, then, by dividing by x, and multiplying the result by n^2, the equation will become $2n^2x+2n^2=m^2x$; and consequently $x=\frac{2n^2}{m^2-2n^2}$.

Where, if n be taken $=2$ and $m=3$, we shall have

$$x=8, \text{ and } \frac{x^2+x}{2}=\frac{64+8}{2}=\frac{72}{2}=36,$$

for the least integral triangular number that is at the same time a square.

3. It is required to divide a given square number into two such parts, that each of them shall be a square number.

Let $a^2=$ the square to be divided, $x^2=$ one of its square parts, and $a^2-x^2=$ the other; which is also to be a square.

Then, agreeably to Case 3, make $a^2-x^2=(nx-a)^2$ $=n^2x^2-2anx+a^2$, and we shall have $2anx=$ $n^2x^2+x^2$, or $n^2x+x=2an$; and consequently $x=$ $\frac{2an}{n^2+1}$, and $nx-a=\frac{2an^2}{n^2+1}-a=\frac{2an^2}{n^2+1}-\frac{an^2+a}{n^2+1}=\frac{an^2-a}{n^2+1}$.

Hence $(\frac{2an}{n^2+1})^2$ and $(\frac{an^2-a}{n^2+1})^2$ are the parts required; where a and n may be any numbers whatever, provided n be greater than 1.

4. It is required to find the least integral number such, that if 4 times its square be added to 29, the result shall be a square.

This being the same thing as to make $4x^2+29$ a square, let $4x^2+29=(2x+n)^2=4x^2+4nx+n^2$, agreeably to Case 4.

Then $4nx+n^2=29$, or $4nx=29-n^2$; and consequently $x=\frac{29-n^2}{4n}$; where, if n be taken equal to 1, we shall have $x=\frac{29-1}{4}=\frac{28}{4}=7$, which is the only integral number that answers the conditions of the question.

5. It is required to find such a value of x that $\sqrt{(6x^2+13x+6)}$ shall be rational, and consequently $6x^2+13x+6$ a square.

Here, if the expression be compared with the general formula $a+bx+cx^2$, we shall have $a=6$, $b=13$, and $c=6$; and as neither a nor c are squares, but $b^2-4ac=169-144=25$ is a square, we are sure that $6x^2+13x+6$ can be represented by two factors; which, upon trial, are found to be $(2x+3)$ and $(3x+2)$.

Let therefore $\sqrt{(6x^2+13x+6)}=\sqrt{\overline{(2x+3)(3x+2)}}$ $=\frac{m}{n}(3x+2)$, as in Case 5.

Then there will arise $(2x+3)\times(3x+2)=\frac{m^2}{n^2}(3x+2)^2$, or $2x+3=\frac{m^2}{n^2}(3x+2)$; and consequently, by reduction, $x=\frac{3n^2-2m^2}{3m^2-2n^2}$.

Where it appears, that, in order to obtain a rational answer, $\frac{m^2}{n^2}$ must be less than $\frac{3}{2}$ and greater than $\frac{2}{3}$.

Whence, if $m=6$ and $n=5$, we shall have $x=\frac{3\times 25-2\times 36}{3\times 36-2\times 25}=\frac{75-72}{108-50}=\frac{3}{58}$, the value required.

6. It is required to find such a value of x that $\sqrt{(13x^2+15x+7)}$ shall be rational, or $13x^2+15x+7$ a square.

Here, if the expression be compared with the general formula $a+bx+cx^2$, it will appear, that as neither a, c, nor b^2-4ac are squares, the question, if possible, can only be resolved according to the method pointed out in Case 6.

In order therefore to try it, in this way, let it be separated into the two parts $(1-x)^2$ and $6+17x+12x^2$.

Then, since $17^2-4(6\times 12)$, which is equal to 1, is a square, the latter part may be divided into two factors, which are $(2+3x)\times(3+4x)$; and consequently the original formula may be represented by $(1-x)^2+(2+3x)\times(3+4x)$.

Hence, putting $\sqrt{(13x^2+15x+7)}=\sqrt{\{(1-x)^2+(2+3x)\times(3+4x)\}}=(1-x)+n(2+3x)$, as in Case 6, we shall have $(1-x)^2+(2+3x)\times(3+4x)=(1-x)^2+2n(1-x)\times(2+3x)+n^2(2+3x)^2$, or $3+4x=2n(1-x)+n^2(2+3x)$; and consequently, by reduction, $x=\frac{2n+2n^2-3}{4+2n-3n^2}$.

Where, taking $n=1$, we have $x=\frac{2+2-3}{4+2-3}=\frac{1}{3}$; and

$$13x^2+15x+7=\frac{13}{9}+\frac{15}{3}+7=\frac{13}{9}+\frac{45}{9}+\frac{63}{9}=\frac{121}{9}$$

a square number as required.

7. It is required to render the formula $\sqrt{(7x^2+2)}$ rational, or to find such a value of x as will make $7x^2+2$ a square.

Here it is soon found, that neither of the former rules will apply; but, as the expression evidently becomes a square when $x=1$, let $x=1+y$, and we shall have $7x^2+2=7y^2+14y+9$; or, by putting $\surd(7y^2+14y+9)=3+\frac{ny}{m}$, and squaring each side,

$$7y^2+14y+9=9+\frac{6ny}{m}+\frac{n^2y^2}{m^2}.$$

Hence, by rejecting the two 9's, and dividing the remaining terms by y, we have $7y+14=\frac{6n}{m}+\frac{n^2y}{m^2}$, or $7m^2y+14m^2=6nm+n^2y$; and consequently $y=\frac{6nm-14m^2}{7m^2-n^2}$, and $x=1+\frac{6nm-14m^2}{7m^2-n^2}=\frac{6nm-7m^2-n^2}{7m^2-n^2}=$

$$\frac{7m^2+n^2-6nm}{n^2-7m^2}.$$

Where it is evident, that m and n may be any positive or negative numbers whatever.

If, for instance, $m=1$ and $n=1$, we shall have $x=-\frac{1}{3}$, or, since the second power of x only is concerned, $x=\frac{1}{3}$, and $7x^2+2=\frac{25}{9}$ a square.

Also, if $n=3$ and $m=-1$, we shall have $x=17$ and $7x^2+2=2025=25^2$, a square number as before; and, by proceeding in this manner, we may obtain as many values of x as we please.

PROBLEM II.

To render surd quantities of the form $\surd(a+bx+cx^2+dx^3)$ rational, or to find such values of x as will make $a+bx+cx^2+dx^3$ a square.

This problem is much more limited, and difficult to be resolved, than the former, there being but a few cases of it that admit of answers in rational numbers; and in these, the rules for obtaining them are of such a confined nature, that when the unknown quantity has more than one value, which, however, is not often the case, the rest can only be determined one at a time, by repeating the operation with the value last obtained, as often as may be found necessary.

Rule.

1. In the case where $a=0$ and $b=0$, put the remaining part $\sqrt{(cx^2+dx^3)}=nx$, or $cx^2+dx^3=n^2x^2$, then we shall have $c+dx=n^2$; and consequently $x=\frac{n^2-c}{d}$.

Where n may be any number whatever greater than the square root of c.

2. When a is a square number put it $=e^2$, and make $\sqrt{(e^2+bx+cx^2+dx^3)}=e+\frac{b}{2e}x$, or $e^2+bx+cx^2+dx^3=(e+\frac{b}{2e}x)^2=e^2+bx+\frac{b^2}{4e^2}x^2$.

Then, as the first two terms on each side of this last equation are equal, we shall have $cx^2+dx^3=\frac{b^2}{4e^2}x^2$, or, by division and reduction, $4de^2x+4ce^2=b^2$; and consequently $x=\frac{b^2-4ce^2}{4de^2}$; or, when $c=0$, $x=\frac{b^2}{4de^2}$.

And if, in the same case, we put $e^2+bx+cx^2+dx^3=(e+\frac{b}{2e}x+\frac{4ce^2-b^2}{8e^3})^2$, there will arise, by squaring, $e^2+bx+cx^2+dx^3=e^2+bx+cx^2+\frac{b(4ce^2-b^2)}{8e^4}x^3+\frac{(4ce^2-b^2)^2}{64e^6}x^4$.

And since the first three terms, on each side of the equation, now destroy each other, we shall have $\frac{(4ce^2-b^2)^2}{64e^6}x^4 + \frac{b(4ce^2-b^2)}{8e^4}x^3 = dx^3$; and consequently, by dividing by x^3, and reducing the result, $x = \frac{64de^6 - 8be^2(4ce^2-b^2)}{(4ce^2-b^2)^2}$, or, if $b=0$, $x = \frac{4de^2}{c^2}$, and, if $c=0$, $x = 8e^2(\frac{8de^4+b^3}{b^4})$ (z).

3. These being all the cases of this formula that admit of any direct rule, it is only necessary to observe, that, if we can find, by trials, as in the former problem, any one value of the unknown quantity that makes the given expression a square, other values of it may be readily determined.

Thus, let p be a value of x, so found, and make $a + bp + cp^2 + dp^3 = q^2$; then by putting $x = y + p$, we shall have $a + bx + cx^2 + dx^3 = a + b(p+y) + c(p+y)^2 + d(p+y)^3 = a + bp + cp^2 + dp^3 + (b + 2cp + 3dp^2)y + (c + 3dp)y^2 + dy^3$, or $a + bx + cx^2 + dx^3 = q^2 + (b + 2cp + 3dp^2)y + (c + 3dp)y^2 + dy^2$; from which latter form the value of y, and consequently that of x, may be found by either of the methods given in Case 2.

It may also be farther remarked, that, if the given formula, in any case of this kind, can be resolved into factors, such that one of them shall be a square, it will be sufficient to make the remaining factor a square in order to render the whole expression so; since a square, multiplied or divided by a square, is still a square.

(z) In the first of these methods the assumed root $e + \frac{b}{2e}x$ is determined by first taking it in the form $e + nx$, and then equating the second term of it, when squared, with the second term of the original formula; in which case n will be found $= \frac{b}{2e}$.

EXAMPLES.

1. It is required to find such a value of x that $\sqrt{(3x^2+11x^3)}$ shall be rational, and consequently $3x^2+11x^3$ a square.

Let $\sqrt{(3x^2+11x^3)}=nx$, or $3x^2+11x^3=n^2x^2$, agreeably to Case 1.

Then, by dividing by x^2, we shall have $3+11x=n^2$.

And consequently $x=\frac{n^2-3}{11}$, where n may be any number, positive or negative, that is greater than $\sqrt{3}$.

Taking, therefore, $n=2, 3, 4, 5$, &c. respectively, we shall have $x=\frac{1}{11}, \frac{6}{11}, \frac{13}{11}, \frac{22}{11}$, or 2, the last of which is the least integral answer that the question admits of.

2. It is required to find such a value of x, that $1+2x-x^2+x^3$ shall be a square.

In like manner, the assumed root $e+\frac{b}{2e}x+\frac{4ce^2-b^2}{8e^3}x^2$, in the second method, is determined by first taking it in the form $e+nx+mx^2$, and then equating the second and third terms of it, when squared, with the second and third terms of the original formula; where n will be found $=\frac{b}{2e}$ and $m=\frac{4ce^2-b^2}{8e^3}$.

The method of determining the factors of which any formula is composed, when it can be done, is to make the given expression $=0$, and then find the roots r, r', &c. of the equation, each of which will give a factor $x-r$, $x-r'$, &c.; and these are generally easily discovered, as we here seek only the rational roots, which are always divisors of the absolute term, or that which does not contain x.

Thus the formula $1-x-x^2+x^3$ is resolvable into the factors $(1-x)\times(1+x)\times(1-x)$, or $(1-x)^2\times(1+x)$; and by putting $1+x=n^2$, we have $x=n^2-1$; where, if n be taken equal to any

Here, 1 being a square, let $1+2x-x^2+x^3=(1+x)^2=1+2x+x^2$, agreeably to the first part of Case 2; then, since the first two terms on each side of the equation destroy each other, we shall have $x^3-x^2=x^2$, or $x^3=2x^2$; and consequently $x=2$, and $1+2x-x^2+x^3=1+4-4+8=9$, a square, as required.

Again, by putting $x=y+3$, according to Case 3, we shall have $1+2x-x^2+x^3=1+2(y+3)-(y+3)^2+(y+3)^3=25+23y+8y^2+y^3$; and making this last exprssion $25+23y+8y^2+y^3=(5+\frac{23}{10}y)^2=25+23y+\frac{529}{100}y^2$, agreeably to the first part of Case 2, there will arise $y^3+8y^2=\frac{529}{100}y^2$, or $y+8=\frac{529}{100}$.

Whence $y=\frac{529}{100}-8=\frac{529-800}{100}=-\frac{271}{100}$, and $x=3-\frac{271}{100}=\frac{300-271}{100}=\frac{29}{100}$ for a second value of x; which, being substituted in the original formula, makes $1+2x-x^2+x^3=\frac{1520289}{1000000}=(\frac{1233}{100})^2$ a square, as was required.

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number whatever, $1-x-x^2+x^3$ will be a square; though, by any other mode of solution, it would be difficult to find even a few values of x.

It may here, also, be observed, that there are but few questions in this problem that can be determined in whole numbers; several of them, likewise, admit only of one answer, and others are totally irresolvable, either in integers or fractions. Thus, if it were required to make $1+x^3$ a square, the only positive value of x that renders this possible, is 2; and the making of $3x^3-1$ a square, is impossible.

3. It is required to find such values of x as will render the formula $4+6x-5x^2+3x^3$ a square.

Here, 4 being a square, let $4+6x-5x^2+3x^3=(2+\frac{3}{2}x)^2=4+6x+\frac{9}{4}x^2$, as in the first part of Case 2; then, as the first two terms on each side of the equation destroy each other, we shall have $3x^3-5x^2=\frac{9}{4}x^2$, or $3x-5=\frac{9}{4}$; and consequently $x=\frac{5+2\frac{1}{4}}{3}=\frac{29}{12}=2\frac{5}{12}$, & $(2+\frac{3}{2}\times\frac{29}{12}x^2)^2=(2+\frac{29}{8})^2=(\frac{45}{8})^2$ a square, as was required.

Or, by the second method of the same case, let $4+6x-5x^2+3x^3=(2+\frac{3}{2}x-\frac{29}{16})^2=4+6x-5x^2-\frac{87}{16}x^3+\frac{841}{256}x^4$; then, as the first three terms on each side of this equation destroy each other, we shall have $\frac{841}{256}x^4-\frac{87}{16}x^3=3x^3$, or $\frac{841}{256}x-\frac{87}{16}=3$, or $841x-1392=768$; and consequently $x=\frac{1392+768}{841}=\frac{2160}{841}$; which is another value of x, that, being substituted in the original formula $4+6x-5x^2+3x^3$, will make it a square.

4. It is required to find such values of x as will make the formula $3+x^3$ a square.

Here it is evident, that the expression is a square when $x=1$; let therefore $x=1+y$, and we shall have $3+x^3=4+3y+3y^2+y^3$; and as the first term of this is a square, make, according to the first method of Case 2, $4+3y+3y^2+y^3=(2+\frac{3}{4}y)^2=4+3y+\frac{9}{16}y^2$; then, because the first two terms of the different sides of the equation destroy each other, we shall have $y^3 + 3y^2 = \frac{9}{16}y^2$, or $y + 3 = \frac{9}{16}$; and consequently $y = \frac{9}{16} - 3 = \frac{9-48}{16} = -\frac{39}{16}$, and $x = 1 - \frac{39}{16} = \frac{16-39}{16} = -\frac{23}{16}$; which is a second value of x.

Again, let $4 + 3y + 3y^2 + y^3 = (2 + \frac{3}{4}y + \frac{39}{64}y^2)^2 = 4 + 3y + 3y^2 + \frac{117}{128}y^3 + \frac{1521}{4096}y^4$; then, as the first three terms on each side of the equation destroy each other, we shall have $\frac{1521}{4096}y^4 + \frac{117}{128}y^3 = y^3$, or $\frac{1521}{4096}y + \frac{117}{128} = 1$; and consequently $y = \frac{352}{1521}$, and $x = 1 + \frac{352}{1521} = \frac{1873}{1521}$, which is a third value of x.

And by proceeding in the same way with either of these new values of x as with the first, other values of it may be obtained; but the resulting fraction will become continually more complex in each operation.

PROBLEM III.

To render surd quantities of the form $\sqrt{(a + bx + cx^2 + dx^3 + ex^4)}$, rational, or to find such values of x as will make $a + bx + cx^2 + dx^3 + ex^4$ a square.

The resolution of expressions of this kind, in which the indeterminate, or unknown quantity, rises to the fourth power, is the utmost limit of the researches that have hitherto been made on formulæ affected by the sign of the square root; and in this problem, as well as in that last given, there are only a few particular cases that admit of answers in rational numbers; the rest being either impossible, or such as afford only one or two simple solutions, which may be found by inspection(*a*).

RULE.

1. In the case where a is a square number, put it $=f^2$, and make $f^2+bx+cx^2+dx^3+ex^4=(f+\frac{b}{2}gx+\frac{4cf^2-b^2}{8f^3}x^2)^2=f^2+bx+cx^2+\frac{b(4cf^2-b^2)}{8f^4}x^3+\frac{(4cf^2-b^2)^2}{64f^6}x^4$; then, since the first three terms on each side of the equation destroy each other, we shall have $\frac{(4cf^2-b^2)^2}{64f^6}x^4+\frac{b(4cf^2-b^2)}{8f^4}x^3=ex^4+dx^3$; and consequently, by dividing by x^3, and reducing the result, $x=\frac{64df^6-8bf^2(4cf^2-b^2)}{(4cf^2-b^2)^2-64ef^6}$; which form fails when any two of the coefficients b, e, d, are each $=0$.

2. When e is a square number, put it $=g^2$, and make $g^2x^4+dx^3+cx^2+bx+a=(gx^2+\frac{d}{2g}x+\frac{4cg^2-d^2}{8g^3})^2$ $=g^2x^4+dx^3+cx^2+\frac{d(4cg^2-d^2)}{8g^4}x+\frac{(4cg^2-d^2)^2}{64g^6}$; then,

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(*a*) It may here be observed, as in the former problem, that some cases of this formula admit only of a limited number of answers; and that others are wholly irresolvable. Thus the only value of x, in the form $2-3x^2+2x^4$, that renders it a square, is 1; also the form $1-x^2+x^4$ can never be a square, except when $x=1$ or -1; and the forms $4x^4-2$ and x^4-x^2+1 are impossible.

Besides the method, given in the text, the formula $a+bx+cx^2+dx^3+g^2x^4$ may be reduced to the preceding case, by putting $x=\frac{1}{y}$; for when $a+\frac{b}{y}+\frac{c}{y^2}+\frac{d}{y^3}+\frac{g^2}{y^4}$ is a square, the product of it by y^4, or $ay^4+by^3+cy^2+dy+g^2$ must also be a square; which is the same as the former expression inverted.

since the first three terms on each side of the equation destroy each other, as before, we shall have $a+bx=\frac{(4cg^2-d^2)^2}{64g^6}+\frac{d(4cg^2-d^2)}{8g^4}x$; and consequently, by transposition and reduction, $x=\frac{(4cg^2-d^2)^2-64ag^6}{64bg^6-8dg^2(4cg^2-d^2)}$; which form also fails in the same case as the former.

3. When the first and last terms are both squares, put $a=f^2$ and $e=g^2$, and make $f^2+bx+cx^2+dx^3+g^2x^4=(f+\frac{b}{2f}x+gx^2)^2=f^2+bx+(2fg+\frac{b^2}{4f^2})x^2+\frac{bg}{f}x^3+g^2x^4$; then, since the second terms, as well as the first and last, on each side of the equation, destroy each other, we shall have $cx^2+dx^3=(2fg+\frac{b^2}{4f^2})x^2+\frac{bg}{f}x^3$; and consequently, by dividing by x^2 and reducing the result, $x=\frac{f^2(c-2fg)-\frac{1}{4}b^2}{f(bg-fd)}$. And because g is found in the original formula only in its second power, it may be taken either negatively or positively; and consequently we shall also have $x=\frac{\frac{1}{4}b^2-f^2(2fg+c)}{f(bg+fd)}$; so that this mode of solution furnishes two different answers.

Or, by making $f^2+bx+cx^2+dx^3+g^2x^4=(f+\frac{d}{2g}x+gx^2)^2=f^2+\frac{df}{g}x+(2fg+\frac{d^2}{4g^2})x^2+dx^3+g^2x^4$, the fourth terms, as well as the first and last, on each side of the equation, will now destroy each other; and consequently we shall have $bx+cx^2=\frac{df}{g}x+(2fg+\frac{d^2}{4g^2})x^2$, or $b+cx=\frac{df}{g}+(2fg+\frac{d^2}{4g^2})x$; where $x=\frac{4g(bg-df)}{4g^2(2fg-c)+d^2}$. And because f is found in the given formula only in its second power, it may be taken either negatively or positively; whence also $x = \frac{4g(bg+df)}{d^2-4g^2(2fg+c)}$; so that this solution likewise furnishes two values of x, each different from the former; but both of them fail in the same case with that before mentioned.

4. When neither the first nor the last terms are squares, the formula cannot be resolved in any other way, than by first endeavouring to discover, by trials, some simple value of the unknown quantity, that will answer the conditions of the question; and then finding other values of it, according to the methods pointed out in the two last problems.

Thus, let p be a value of x, so determined; and make $a + bp + cp^2 + dp^3 + ep^4 = q^2$; then, putting $x = p + y$, we shall have $a + bx + cx^2 + dx^3 + ex^4 = a + b(p+y) + c(p+y)^2 + d(p+y)^3 + e(p+y)^4 = a + bp + cp^2 + dp^3 + ep^4 + by + 2cpy + cy^2 + 3dp^2y + 3dpy^2 + dy^3 + 4ep^3y + 6ep^2y^2 + 4epy^3 + ey^4 = q^2 + (b + 2cp + 3dp^2 + 4ep^3)y + (c + 3dp + 6ep^2)y^2 + (d + 4ep)y^3 + ey^4$; from which latter form, the value of y, and consequently that of x, may be found as in Case 1 (*b*).

(*b*) It may be farther remarked, that the formula $f^2 + bx + cx^2 + dx^3 + g^2x^4$, here treated of, may also be resolved by the methods used in Case 1 and Case 2; so that, if these be taken in conjunction with the two orders of solution last given, the expression will be found to admit of six different answers. The form $a + ex^4$, also, which is one of those that most frequently occur, may be made a square, when a single case is known, by a peculiar artifice of substitution, that will be often found of considerable utility in other instances; which is by putting the value first found $= +p$ or $-p$, or $a + ep^4 = q^2$, and making $x = \frac{p(1+y)}{1-y}$; by which means the formula will become

EXAMPLES.

1. It is required to find such a value of x, that $1-2x+3x^2-4x^3+5x^4$ shall be a square.

Here, the first term being a square number, let $1-2x+3x^2-4x^3+5x^4=(1-x-x^2)^2=1-2x+3x^2-2x^3+x^4$, agreeably to the method in Case 1.

Then, since the first three terms on each side of the equation destroy each other, we shall have $5x^4-4x^3=x^4-2x^3$, and, by dividing by x^3, there will arise $5x-4=x-2$: whence, by transposition, $5x-x=4-2$, or $x=\frac{2}{4}=\frac{1}{2}$; and consequently $1-2x+3x^2-4x^3+5x^4=1-1+\frac{3}{4}-\frac{1}{2}+\frac{5}{16}=\frac{9}{16}$; which is a square number, as was required.

2. It is required to find such a value of x, that $-2+3x-x^2-2x^3+4x^4$ shall be a square.

Here the last term being a square, let $4x^4-2x^3-x^2+3x-2=(2x^2-\frac{1}{2}x-\frac{5}{16})^2=4x^4-2x^3-x^2+\frac{5}{16}x+\frac{25}{256}$, according to the method in Case 2.

Then, since the first three terms on each side of the equation destroy each other, we shall have $3x-2=\frac{5}{16}x+\frac{25}{256}$, or $3x-\frac{5}{16}x=2+\frac{25}{256}$; whence, by

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$$\frac{a(1-y)^4+ep^4(1+y)^4}{(1-y)^4}, \text{ or}$$
$$\frac{q^2+4(q^2-2a)y+6q^2y^2+4(q^2-2a)y^3+q^2y^4}{(1-y)^4};$$
where, as the first and last terms of the numerator, as well as the denominator, are now squares, other values of x may be determined by the methods employed in Case 3.

reduction, $\frac{43x}{16}=\frac{537}{256}$; and, consequently, $x=\frac{537}{256}\times\frac{16}{43}=\frac{537}{16\times 43}=\frac{537}{688}$.

Or, if we put $x=\frac{1}{y}$, the formula will become $\frac{4}{y^4}-\frac{2}{y^3}-\frac{1}{y^2}+\frac{3}{y}-2$; and, multiplying this by y^4, which is a square, it will be $4-2y-y^2+3y^3-2y^4$; where the first term being now a square, if the expression, so transformed, be resolved by Case 1, we shall have $y=\frac{688}{537}$; and consequently $x=\frac{1}{y}=\frac{537}{688}$, as before.

3. It is required to find such values of x, as shall make $1+3x+7x^2-2x^3+4x^4$ a square.

Here, both the first and last terms being squares, let $1+3x+7x^2-2x^3+4x^4=(1+\frac{3}{2}x+2x^2)^2=1+3x+\frac{25}{4}x^2+6x^3+4x^4$, according to the method in Case 3.

Then, since the second terms, as well as the first and last, on each side of the equation, destroy each other, we shall have $6x^3+\frac{25}{4}x^2=7x^2-2x^3$; or $6x+2x=7-\frac{25}{4}=\frac{3}{4}$: whence, by reduction, $x=\frac{3}{32}$.

And, if we put the same formula $1+3x+7x^2-2x^3+4x^4=(1+\frac{3}{2}x-2x^2)^2,=1+3x-\frac{7}{4}x^2-6x^3+4x^2$, there will arise, by cancelling the equal terms on each side of the equation, $7x^2-2x^3=-\frac{7}{4}x^2-6x^3$; and, by division and transformation, $6x-2x=-\frac{7}{4}-7=-\frac{35}{4}$; or $x=-\frac{35}{16}$.

And, in a similar manner, other values of x may be found, by employing the method of substitution pointed out in the latter part of Case 3.

4. It is required to find such values of x as will make $2x^4 - 1$ a square.

Here, by Case 4, as 1 is an obvious value of x, let $x = 1 + y$.

Then $2x^4 - 1 = 2(1 + y)^4 - 1 = 2(1 + 4y + 6y^2 + 4y^3 + y^4) - 1 = 1 + 8y + 12y^2 + 8y^3 + 2y^4$.

And since the first term of this last expression is now a square, we shall have, by Case 1, $1 + 8y + 12y^2 + 8y^3 + 2y^4 = (1 + 4y - 2y^2)^2 = 1 + 8y + 12y^2 - 16y^3 + 4y^4$; and as the three first terms of the two members of this equation destroy each other, there will remain $4y^4 - 16y^3 = 2y^4 + 8y^3$. Or, dividing by x^3, and reducing the result, $y = 12$; and consequently $x = 1 + y = 13$. Whence $2x^4 - 1 = 57121 = (239)^2$.

And if 13 be now taken, as the known value of x, and the operation be repeated as before, we shall obtain, for another value of x, the complicated fraction $\frac{10607469769}{2447192159}$ (c).

PROBLEM IV.

To render surd quantities of the form $\sqrt[3]{(a + bx +}$

(c) Formulæ of the above kind, which contain the fifth, or other higher powers of x, cannot be resolved by any artifices that have yet been discovered, even though a single case should be known; for if we put the expression $g^2 + bx + cx^2 + dx^3 + ex^4 + fx^5 = (g + px + qx^2)^2$, and determine p and q in the expansion of the latter, so as to make the second and third terms vanish, there will still remain three terms, which, when divided by x^3, form a quadratic equation; so that x cannot be determined, except by means of an irrational quantity. And if we were to suppose the

cx^2+dx^3) rational, or to find such values of x as will make $a+bx+cx^2+dx^3$ a cube.

This formula, like the two latter of those relating to squares, cannot be resolved by any direct method, except in the cases where the first or last terms of the expression are cubes; it being necessary, in all the rest, that some simple number, answering the conditions of the question, should be first known, before we can hope to obtain others; which answer, if it does not readily occur, must be found by substituting for the unknown quantity such small integers or fractions, as appear most likely to suit the purpose intended (*d*).

RULE.

1. In the case where a is a cube number, put it $=e^3$, and take $e^3+bx+cx^2+dx^3=(e+\frac{b}{3e^2}x)^3=e^3+bx+\frac{b^2}{3e^3}x^2+\frac{b^3}{27e^6}x^3$; then, since the two first terms on each side of the equation are equal to each other,

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root $=f+px+qx^2+rx^3$, its square would rise to the 6th power; and, consequently, if p, q, and r could be determined, so as to remove the 2d, 3d, and 4th terms, there would still remain the 4th, 5th, and 6th powers; which, being divided by x^4, would again leave a quadratic equation: hence it may be inferred, that the converting of certain formulæ into squares, by the methods already delivered, has been completely exhausted.

(*d*) It may here be observed, that Euler, Legendre, and other writers on the Diophantine Algebra, have demonstrated, that neither the sum nor the difference of any two cubes can be a cube, or the double of a cube; and that neither the sum nor the difference of two biquadrates can be either a square or a biquadrate; which are, indeed, only particular cases of Fermat's genewe shall have $dx^3 + cx^2 = \frac{b^3}{27e^6}x^3 + \frac{b^2}{3e^3}x^2$; or, by dividing by x^2, and reducing the terms, $27de^6x + 27ce^6 = b^3x + 9b^2e^3$: whence $x = \frac{9e^3(3ce^3 - b^2)}{b^3 - 27de^6}$.

2. When d is a cube number, put it $= f^3$, and take $a + bx + cx^2 + f^3x^3 = (\frac{c}{3f^2} + fx)^3 = \frac{c^3}{27f^6} + \frac{c^2}{3f^3}x + cx^2 + f^3x^3$; then since the two last terms on each side of the equation now destroy each other, we shall have $a + bx = \frac{c^3}{27f^6} + \frac{c^2}{3f^3}x$; and consequently, by reduction, $x = \frac{27af^6 - c^3}{9f^3(c^2 - 3bf^3)}$.

3. When a and d are both cube numbers, let them be put $= e^3$ and f^3; and make $e^3 + bx + cx^2 + f^3x^3 = (e + fx)^3 = e^3 + 3fe^2x + 3ef^2x^2 + f^3x^3$; then, since the first and last terms of the equation now destroy each other, we shall have $bx + cx^2 = 3fe^2x + 3ef^2x^2$; and consequently, by dividing by x, and transposing the terms, $x = \frac{3fe^2 - b}{c - 3ef^2}$; which formula may also be resolved by either of the two first cases.

4. When neither the first term nor the last are cubes, let p be a value of x found by inspection, or

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ral theorem; who, in his observations on the questions of Diophantus, asserts, that the equation $x^n \pm y^n = z^n$, is always impossible for every integral value of n, that is greater than 2: but this was left by him without proof, and has never yet been completely demonstrated by any succeeding writer.

But it is easy to resolve, without much trouble, another question of a similar kind, which is to find three cubes, that, when taken together, shall make a cube; the least whole numbers of this kind being 3, 4, and 5, the sum of the cubes of which is $3^3 + 4^3 + 5^3 = 216 = 6^3$.

by trials, and make $a+bp+cp^2+dp^3=q^3$; then putting $x=p+y$, we shall have $a+bx+cx^2+dx^3=a+b(p+y)+c(p+y)^2+d(p+y)^3=a+bp+cp^2+dp^3+(b+2cp+3dp^2)y+(c+3dp)y^2+dy^3=q^3+(b+2cp+3dp^2)y+(c+3dp)y^2+dy^3$; from which latter form, as the first term is now a cube, the value of y, and consequently that of x, may be found as in Case 1.

EXAMPLES.

1. It is required to find such a value of x as will make the formula $1+x+x^2$ a cube.

Here, the first term being a cube, let the root $=1+\frac{1}{3}x$, according to Case 1; then we shall have $1+x+x^2=1+x+\frac{1}{3}x^2+\frac{1}{27}x^3$; and, since the two first terms of this equation destroy each other, there will remain $x^2=\frac{1}{3}x^2+\frac{1}{27}x^3$, which, being divided by x^2, gives $\frac{1}{27}x+\frac{1}{3}=1$, or, by reduction, $x+9=27$; whence $x=27-9=18$; and consequently $1+x+x^2=1+18+324=343=7^3$, a cube number, as was required. And if we now take $x=18$, and proceed according to the method employed in Case 4, we shall obtain $x=-\frac{137826}{50653}$; which will, also, lead, in like manner, to other new values.

2. It is required to find such a value of x as will make $133+3x^2+x^3$ a cube.

Here, the last term being a cube, let its root $=1+x$, according to Case 2; then we shall have $133+3x^2+x^3=(1+x)^3=1+3x+3x^2+x^3$; and since the two last terms of this equation destroy each other; there will remain $1+3x=133$, or $3x=133-1=132$; whence $x=\frac{132}{3}=44$, and consequently $133+3x^2+x^3=91125=(45)^3$, a cube number, as was required.

And if 45 be now taken as a known value of x, other values of it may be found, as in the last example.

3. It is required to find such a value of x, that $8+28x+89x^2-125x^3$ shall be a cube.

Here, both the first and last terms being cubes, let the root $=2-5x$, according to Case 3; then we shall have $8+28x+89x^2-125x^3=(2-5x)^3=8-60x+150x^2-125x^3$; and, since the first and last terms of this equation destroy each other, there will remain $28x+89x^2=-60x+150x^2$; and, by dividing by x, and transposing the terms, $150x-89x=28+60$, or $61x=88$; whence $x=\frac{88}{61}$, the value required. And as this formula can, also, be resolved either by the first or second case, other values of x may be obtained, that will equally answer the conditions of the question.

4. It is required to find such a value of x, that the formula $3+3x^3$ shall be a cube.

Here, 2 being a value of x, that is readily found by inspection, let, by Case 4, $x=2+y$; then we shall have $3+3x^3=3+3(2+y)^3=27+36y+18y^2+3y^3$; and as the first term of this latter expression is now a cube, let its root $=3+\frac{4}{3}y$, according to Case 1; then there will arise $27+36y+18y^2+3y^3=27+36y+16y^2+\frac{64}{27}y^3$; or, since the two first terms of the equation destroy each other, $18y^2+3y^3=16y^2+\frac{64}{27}y^3$; and, by dividing by y^2, and reducing the terms, $y=-\frac{54}{17}$; whence $x=2-\frac{54}{17}=\frac{34-54}{17}=-\frac{20}{17}$; and consequently $3+3x^3=3-\frac{24000}{4913}=\frac{14739-24000}{4913}=-\frac{9261}{4913}=-(\frac{21}{17})^3$ a cube number, as was required (*e*).

PROBLEM V.

Of the resolution of double and triple equalities.

When a single formula, containing one or more unknown quantities, is to be transformed to a perfect power, such as a square or a cube, this is called, in the Diophantine Analysis, a simple equality; and when two formulæ, containing the same unknown quantity, or quantities, are to be each transformed to some perfect power, it is then called a double equality, and so on; the methods of resolving which, in such cases as admit of any direct rule, are as follows.

(*e*) It may here be observed, that any simple formula of the kind ax^2 may be readily transformed into a cube, in a general manner, by putting its root $=\frac{x}{p}$; in which case we have $ax^2=\frac{x^3}{p^3}$, or $x=ap^3$; where p may be any number whatever.

Also any expression of the form $a(b+cx)^2$ may be readily made a cube, by putting its root $=\frac{b+cx}{p}$; for, since in this case $a(b+cx)^2=\frac{(b+cx)^3}{p^3}$, there will arise, by division, $b+cx=ap^3$, or $x=\frac{ap^3-b}{c}$, where p is arbitrary, as before: both of which cases

RULE.

1. In the case where the unknown quantity does not exceed the first degree, as in the double equality $a+bx=$ a square, and $c+dx=$ a square, let the first of these formulæ $a+bx=t^2$, and the second $c+dx=u^2$; then, by expunging x from each of these equations, we shall have $bu^2+ad-bc=dt^2$, or $bdu^2+(ad-bc)d=d^2t^2$; and since the quantity on the right hand side of this last equation is now a square, it is only necessary to find such a rational value of u, as will make $bdu^2+(ad-bc)d$ a square; which being done according to one of the methods already explained, we shall have $x=\frac{u^2-c}{d}$.

2. When the unknown quantity does not exceed the second degree, and is found in all the terms of the two formulæ, as in the double equality $ax^2+bx=$ a square, and $cx^2+dx=$ a square, let $x=\frac{1}{y}$; then, by multiplying each of the two resulting expressions by y^2, we shall have $a+by=$ a square, and $c+dy=$ a square; from which the value of y,

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show how useful it is, in this branch of analysis, to reduce a given formula into its factors, as often as it can be done.

In addition to this, it may likewise be remarked, that any formula, in which the unknown quantity does not exceed the second power, may be transformed into a biquadrate, when the case is possible, by first making it a square, and then transforming the root of this square into another square, by the rules before given. Thus, if it were required to make $7+x^2$ a biquadrate, we should have, by pursuing the method here proposed, $x=\frac{23}{12}$.

and consequently that of x, may be determined as in Case 1.

But if it were required to transform the two general expressions $a+bx+cx^2$ and $d+ex+fx^2$ into squares, the solution could only be obtained in a few particular cases, as the resulting equality would rise to the fourth power.

3. In the case of a triple equality, where three expressions of the form $ax+by$, $cx+dy$, and $ex+fy$, are to be transformed to squares, let the first of them $ax+by=t^2$, the second $cx+dy=u^2$, and the third $ex+fy=s^2$; then, if x be expunged from each of these equations, we shall have $(af-be)u^2-(cf-de)t^2=(ad-cb)s^2$; and by putting $\frac{u}{t}=z$, or $u=tz$, there will arise $\frac{af-be}{ad-cb}z^2-\frac{cf-de}{ad-cb}=\frac{s^2}{t^2}$; and since the quantity on the right hand side of this equation is a square, it only remains to find such a rational value of z as will make $\frac{af-be}{ad-cb}z^2-\frac{cf-de}{ad-cb}$ a square; which being done by one of the methods before explained, we shall readily obtain, by means of the first two equations, $x=\frac{d-bz^2}{ad-cb}t^2$, and $y=\frac{az^2-c}{ad-cb}t^2$, where t may be any number whatever.

But if the three formulæ, here proposed, contained only one variable quantity, the simple equality, to which it would be necessary to reduce them, would rise, as in the last case, to the fourth power; and be equally limited with respect to its solution.

4. In other cases of this kind, all that can be done is to find successively, by the former rules,

several answers, when one is known; and, if neither this nor any of the above mentioned modes of solution are found to succeed, the problem under consideration can only be determined by adopting some artifice of substitution, that will fulfil one or more of the required conditions, and then resolving the remaining formulæ, when they are possible, by the methods already delivered for that purpose; but as no general precepts can be given, for obtaining the solution in this way, the proper mode of proceeding, in such cases, must chiefly depend upon the skill and sagacity of the learner (*f*).

EXAMPLES.

1. It is required to find a number x such, that $x+128$ and $x+192$ shall be both squares.

Here, according to Case 1, let $x+128=u^2$, and $x+192=t^2$; then, by expunging x, we shall have $u^2-128=t^2-192$, or $u^2+64=t^2$; and, as the quantity on the right hand side of the equation is now a square, it only remains to make u^2+64 a square; for which purpose put its root $=u+n$; then $u^2+64=u^2+2nu+n^2$, or $2nu+n^2=64$; whence $u=\frac{64-n^2}{2n}$: or, taking n, which is arbitrary, $=2$, we

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(*f*) Many expressions that are susceptible of being made complete squares, or cubes, when taken singly, are often far otherwise when taken in pairs: thus, the equations $x^2+y^2=z^2$, and $x^2-y^2=w^2$; or $x^2+y^2=z^2$, and $x^3-3y^2=w^2$, are impossible; there being no whole or fractional numbers whatever, that, when substituted for x and y, will make each of these double equalities squares.

shall have $u=\frac{64-4}{4}=\frac{60}{4}=15$; and consequently $x=u^2-128=15^2-128=225-128=97$, the answer.

2. It is required to find a number x such, that x^2+x and x^2-x shall be both squares.

Here, according to Case 2, let $x=\frac{1}{y}$; then the two formulæ in the question will become $\frac{1}{y^2}+\frac{1}{y}$ and $\frac{1}{y^2}-\frac{1}{y}$, or $\frac{1}{y^2}(1+y)$ and $\frac{1}{y^2}(1-y)$, which are to be squares; but since a square number, when multiplied or divided by a square number, is still a square, it is the same thing as to transform $1+y$ and $1-y$ to squares; for which purpose, let $1+y=p^2$, or $y=p^2-1$; then $1-y=2-p^2$, which is also to be a square. But as neither the first nor last terms of this new formula are squares, we must, in order to succeed, find some simple number, that will answer the condition required; which it is evident, from inspection, will be the case when $p=1$.

Let, therefore, $p=1-q$, and we shall have $1-y=2-p^2=1+2q-q^2$; or putting $1-rq$ for the root of the latter expression, $1+2q-q^2=1-2rq+r^2q^2$; whence $2-q=-2r+r^2q$, or $q=\frac{2r+2}{r^2+1}$; and consequently $x=\frac{1}{y}=\frac{1}{q^2-2q}=\frac{(1+r^2)^2}{4r-4r^3}$, where, in order to render x positive, r may be taken equal to any proper fraction whatever.

Let, therefore, for the sake of greater simplicity, $r=\frac{t}{u}$, and we shall then have $x=\frac{(u^2+t^2)^2}{4tu(u^2-t^2)}$; in which case any whole numbers may now be substituted for u and t, provided u be made greater than t.

If, for instance, $u=2$ and $t=1$, we shall have $x=\frac{25}{24}$; and if $u=3$ and $t=2$, $x=\frac{169}{120}$; and so on for any other numbers.

3. It is required to find three numbers in arithmetical progression such, that the sum of every two of them may be a square.

Let x, $x+y$, and $x+2y$, = the three numbers; and put $2x+y=t^2$, $2x+2y=u^2$, and $2x+3y=s^2$, agreeably to Case 3; then, by expunging x and y from these equations, we shall have $u^2-t^2=s^2-u^2$, or $2u^2-t^2=s^2$; and if we now put $u=tz$, there will arise $2t^2z^2-t^2=s^2$, or $2z^2-1=\frac{s^2}{t^2}$; where, $\frac{s^2}{t^2}$ being a square, it only remains to make $2z^2-1$ a square; which it evidently is when $z=1$. But as this value would be found not to answer the conditions of the question, let $z=1-p$; then $2z^2-1=2(1-p)^2-1=1-4p+2p^2$; and by putting this last expression $=(1-rp)^2$, we have $1-4p+2p^2=1-2rp+r^2p^2$, or $-4+2p=-2r+r^2p$; whence $p=\frac{2r-4}{r^2-2}$, and $z=1-\frac{2r-4}{r^2-2}=\frac{r^2-2r+2}{r^2-2}$; or, making $r=\frac{m}{n}$, $z=\frac{m^2-2mn+2n^2}{m^2-2n^2}$; and since, by the first two equations, $y=u^2-t^2=t^2z^2-t^2=(z^2-1)t^2$, and $x=\frac{1}{2}(t^2-y^2)=\frac{1}{2}(2-z^2)t^2$, it is evident, that z must be some number greater than 1 and less than $\sqrt{2}$.

If, therefore, $m=9$ and $n=5$ we shall have $z=\frac{81-90+50}{81-50}=\frac{41}{31}$, $x=\frac{241}{31^2}\times\frac{t^2}{2}$, and $y=\frac{720}{31^2}\times t^2$; or, taking $t=2\times 31$, $x=482$, and $y=2880$. Hence

$x=482$, $x+y=3362$, and $x+2y=6242$, the numbers required.

4. It is required to divide a given square number into two such parts, that each of them shall be a square (g).

Let $a^2=$ given square number, and x^2 and a^2-x^2 its two parts; then, since x^2 is a square, it only remains to make a^2-x^2 a square; for which purpose let its root $=rx-a$, and we shall have $a^2-x^2=r^2x^2-2arx+a^2$, or $-x^2=r^2x^2-2arx$; whence, by reduction, $x=\frac{2ar}{r^2+1}$, the root of the first part, and $rx-a=\frac{2ar^2}{r^2+1}-a, =\frac{2ar^2}{r^2+1}-\frac{ar^2+a}{r^2+1}=\frac{ar^2-a}{r^2+1}$, the root of the second.

Therefore $(\frac{2ar}{r^2+1})^2$ and $(\frac{ar^2-a}{r^2+1})^2$ are the parts required; where r may be any number, taken at pleasure, provided it be greater than 1.

5. It is required to divide a given number, consisting of two known square numbers, into two other square numbers.

Let a^2+b^2 be the given number, and x^2, y^2, the two required numbers, whose sum, x^2+y^2, is to be equal to a^2+b^2; then, it is evident, that, if x be either greater or less than a, y will be, accordingly, less or greater than b. If, therefore, we take $x=a+pz$, and $y=b-qz$, there will arise $a^2+2apz+p^2z^2+b^2-2bqz+q^2z^2=a^2+b^2$; or, by reduction, $p^2z+q^2z=2bq-2ap$; whence $z=\frac{2bq-2ap}{p^2+q^2}$; & consequently $x=\frac{2bpq+a(q^2-p^2)}{p^2+q^2}$, & $y=\frac{2apq+b(p^2-q^2)}{p^2+q^2}$; where p and q may be any numbers, taken at pleasure, provided the proportion of them be not the same as a to b, or $a+b$ to $a-b$; as the squares sought would, in this case, come out the same as the known squares; and consequently the operation would be nugatory.

(g) It may here be remarked, as a useful property in this kind of analysis, that if r and s be any two unequal numbers, of which r is the greater, then will $2rs$, r^2-s^2, and r^2+s^2, be the perpendicular, base, and hypothenuse of a right angled triangle. And, from this, two square numbers may be readily found, whose sum, or difference, shall be square numbers. Thus, $(2rs)^2+(r^2-s^2)^2=(r^2+s^2)^2$, and $(r^2+s^2)^2-(2rs)^2=(r^2-s^2)^2$, or $(r^2+s^2)^2-(r^2-s^2)^2=(2rs)^2$; where r and s may be any numbers taken at pleasure.

6. It is required to find two square numbers such, that their difference shall be equal to a given number.

Let d = the given difference, which resolve into two factors a, b, of which a is the greater and b the less; then, putting x for the side of the less square, and $x+b$=side of the greater, we shall have $(x+b)^2-x^2=x^2+2bx+b^2-x^2=d(ab)$, or, by dividing each side by b, $2x+b=a$; whence $x=\frac{a-b}{2}$, the side of the less square sought, and $x+b=\frac{a-b}{2}+b=\frac{a+b}{2}$, the side of the greater. If, for instance, $d=60$, take $a\times b=30\times 2$, and we shall have $x=\frac{30-2}{2}=14$, and $x+2=\frac{30+2}{2}=16$; or $16^2-14^2=256-196=60$, the given difference (h).

(h) This question, though readily resolved according to the method above given, is considered by Diophantus as a very important one, being made the foundation of many of his other

7. It is required to find two numbers such, that, if either of them be added to the square of the other, the sums shall be squares.

Let x and y be the numbers sought; and consequently x^2+y and y^2+x the expressions that are to be transformed to squares.

Then, if $r-x$ be assumed for the side of the first square, we shall have $x^2+y=r^2-2rx+x^2$, or, by reduction, $x=\frac{r^2-y}{2r}$; and if $s+y$ be taken for the side of the second square, we shall have $y^2+\frac{r^2-y}{2r}=s^2+2sy+y^2$; or, by reduction, $r^2-y=4rsy+2rs^2$; whence $y=\frac{r^2-2rs^2}{4rs+1}$, and $x=\frac{2rs^2+s^2}{4rs+1}$, the numbers required; where r and s may be any numbers, taken at pleasure, provided r be greater than $2s^2$.

8. It is required to find two numbers such, that their sum and difference shall be both squares.

Let x and x^2-x be the two numbers sought; then, since their sum is evidently a square, it only remains to make their difference, x^2-2x, a square.

For this purpose, therefore, put its root $=x-r$, and we shall have $x^2-2x=x^2-2rx+r^2$, or, by transposition, $2rx-2x=r^2$; whence $x=\frac{r^2}{2r-2}$, and $x^2-2x=\frac{1}{4}(\frac{r^2}{r-1})^2-\frac{r^2}{r-1}$; where r may be any num-

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problems. The excellent old Kersey, after amplifying and illustrating it in a variety of ways, concludes his chapter thus: "For a farther account of this rare speculation, see Andersonus, theorem 2, of Vieta's mysterious doctrine of Angular Sections; and likewise Herigonius, at the latter end of the first tome of his *Cursus Mathematicus.*"

ber taken at pleasure provided it be greater than 1.

9. It is required to find three numbers such, that not only the sum of all three of them, but also the sum of every two, shall be a square number.

Let $4x$, x^2-4x, and $2x+1$, be the three numbers sought; then $4x+(x^2-4x)=x^2$, $(x^2-4x)+(2x+1)=x^2-2x+1$, & $4x+(x^2-4x)+(2x+1)=x^2+2x+1$, being all evidently squares, it only remains to make the quantity $4x+(2x+1)$, or its equal, $6x+1$, a square; for which purpose, let $6x+1=a^2$, and we shall have $x=\frac{a^2-1}{6}$; whence $\frac{4a^2-4}{6}$, $(\frac{a^2-1}{6})^2-\frac{4a-4}{6}$, and $\frac{2a^2-2}{6}+1$, or their equals, $\frac{2a^2-4}{3}$, $\frac{a^4-26a^2+25}{36}$, and $\frac{a^2+2}{3}$, are the numbers required; where a may be any number taken at pleasure, provided it be greater than 5.

QUESTIONS FOR PRACTICE.

1. It is required to find a number such, that $x+1$ and $x-1$ shall be both squares.

Ans. $x=\frac{5}{4}$

2. It is required to find a number x such, that $x+4$ and $x+7$ shall both be squares.

Ans. $x=\frac{57}{16}$

3. It is required to find two numbers such, that, if their product be added to the sum of their squares, the result shall be a square.

Ans. 3 and 5

4. Find two numbers such, that, if the square of each be added to their product, the sums shall be both squares. Ans. 9 and 16

5. It is required to find two whole numbers such, that the sum or difference of their squares, when diminished by unity, shall be a square.

Ans. 8 and 9

6. To find two whole numbers such, that, if unity be added to each of them, and also to their halves, the sums, in both cases, shall be squares.

Ans. 48 and 1680

7. It is required to find three square numbers, that shall be in arithmetical progression.

Ans. 1, 25, and 49

8. It is required to find three square numbers, that shall be in harmonical proportion.

Ans. 1225, 49, and 25

9. To find three whole numbers such, that, if to the square of each the product of the other two be added, the sums shall be squares.

Ans. 9, 73, and 328

10. To find three numbers in geometrical progression such, that the difference of every two of them shall be a square number.

Ans. 567, 1008, and 1792

11. To find three numbers such, that, if each of them be added to the product of the other two, the sums shall be all squares. Ans. 1, 7, and 9

12. It is required to resolve 4225, which is the square of 65, into two other integral squares.

Ans. 2704 and 1521

13. It is required to resolve 9^2+2^2, or 85, into two other integral squares. Ans. 7^2+6^2

14. It is required to find three square numbers such, that their sum shall be a square.

Ans. 9, 16, and $\frac{144}{25}$

15. To find three numbers such, that their sum and their three differences shall be all squares.

Ans. 24, 168, and 249

16. To find three numbers in geometrical progression such, that, if the mean be added to each of the extremes, the sum, in both cases, shall be squares. Ans. 5, 20, and 80

17. To find three square numbers such, that the sum of every two of them shall be squares.

Ans. 44^2, 117^2, and 240^2

18. To find two integral numbers such, that if unity be added to each, as also to their sum and difference, the four results shall be squares.

Ans. 1368, 840, or 2208 and 528

19. To find three square numbers such, that the differences of every two of them shall be squares.

Ans. 153^2, 185^2, and 697^2

20. To find three numbers such, that the sum, or difference, of any two of them shall be a square number. Ans. 434657, 420968, and 150568

21. It is required to find three square numbers such, that the difference between every two of them and the third shall be a square number.

Ans. 149^2, 241^2, and 269^2

22. To find two numbers such, that their sum shall be equal to the sum of their cubes.

Ans. $\frac{5}{7}$ and $\frac{8}{7}$

23. To find three numbers such, that, if each of them be added to the cube of their sum, the three results shall be all cubes.

Ans. $\frac{23625}{157464}$, $\frac{1538}{157464}$, and $\frac{18577}{157464}$

24. To find three numbers such, that, if each of

them be subtracted from the cube of their sum, the three remainders shall be all cubes.

Ans. $\frac{13851}{85184}$, $\frac{19476}{85184}$, and $\frac{18954}{85184}$(*i*).

OF CONTINUED FRACTIONS.

(H) A CONTINUED FRACTION is that which has for its denominator a whole number and a fraction; and which latter fraction has, also, for its denominator, a whole number and a fraction; and so on, continually, or till the series terminates, by being broken off, after a certain number of terms.

$$\text{Thus, } \cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{6+\cfrac{1}{8\,\&c.}}}} \qquad \text{Or } \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d\,\&c.}}}}$$

(*i*) The two last questions, here given, are reckoned among the most difficult of any of those that have been proposed by Diophantus; and, if solved by his method, the operation will be found extremely intricate and laborious.

Vieta was of opinion, that a number, composed of two known cubes, could not be resolved into any other two cubes; but Fermat, in his observations on the questions of Diophantus, has pointed out a method by which such cubes may be determined; though the calculation, indeed, extends to numbers that are exceedingly complex. Thus, in the simple case of dividing 9, which is the sum of the two cubes 8 and 1, into two other cubes, Pére de Billy, in his *Diophantus Redevivus*, has found, by following the rule laid down by Fermat, that the sides of the two new cubes, answering the conditions required, are

$$\frac{12436177733900097836481}{609623835661372297449} \text{ and } \frac{487267171714352336560}{609623835661372297449};$$

but few, it is believed, will be inclined to ascertain whether the answer, here given, be true or false.

are continued fractions, of the kind that most usually occur (*k*).

These expressions arise in various ways, and are of considerable use in some of the more difficult branches of analysis; particularly in the solution of certain unlimited problems, of the second degree, where the answers, in whole numbers, cannot be easily obtained by any other method.

But one of the greatest practical advantages these formulæ afford, is in approximating to the values of fractions and ratios, that are expressed in large numbers; which is done by dividing the greater of the two numbers by the less, and the last divisor continually by the last remainder, till nothing remains, as in finding their greatest common measure; in which case, the several quotients, thus obtained, are the denominators of the fractions, and their numerators are always 1, or unity.

1. Thus, to represent the common fraction, or ratio, $\frac{a}{b}$, by a continued fraction, let b be contained in a, p times, with a remainder c; also, let c be contained in b, q times, with a remainder d; d in c, r

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(*k*) Lord Brounker appears to have been the first who considered the doctrine of continued fractions, or at least who applied them to the quadrature of curves, and other analytical investigations. (See Wallis's Arith. Infin. vol. i, p. 469.) Huygens also used them in his Descrip. Autom. Planet. for the purpose of approximating to the ratios of large numbers; and since that time they have been amply treated of by various other writers, particularly by Euler, in his Analys. Infin., and by Legendre, in his Essai sur la Théorie des Nombres, second edition.

times, with a remainder e; and so on, as in the following operation:

$$\begin{array}{l}
b)a(p \\ \hline
\quad c)b(q \\ \hline
\qquad d)c(r \\ \hline
\qquad\quad e)d(s \\ \hline
\qquad\qquad f\ \&c.
\end{array}$$

Then we shall have, by the common rule for the division of numbers,

$$\frac{a}{b}=p+\frac{c}{b},\ \frac{b}{c}=q+\frac{d}{c},\ \frac{c}{d}=r+\frac{e}{d},\ \&c.$$

Where the integers p, q, r, &c. are called quotients; and each of these, with its depending fraction, *complete quotients.*

And if b be again divided by c, c by d, d by e, &c. there will arise, by the successive substitution of the results,

$$\frac{a}{b}=p+\frac{c}{b}=p+\cfrac{1}{q+\cfrac{d}{c}}=p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{e}{d}}}\ \&c.$$

Whence, by extending the number of terms, and generalizing the formula, we shall have

$$\frac{a}{b}=p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}}}\ \&c. \qquad \text{Or } \frac{a}{b}=\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}}}}\ \&c.$$

according as the numerator is greater or less than the denominator: which expressions, in this case, will always consist of a greater or less number of terms, according as the fraction $\frac{a}{b}$ is more or less complex.

Thus, if it be required to reduce $\frac{1393}{972}$ to a continued fraction, we shall have

$$\frac{1393}{972} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{6}}}}}$$

And if $\frac{1171}{9743}$ is to be reduced to a continued fraction, we shall have

$$\frac{1171}{9743} = \cfrac{1}{8 + \cfrac{1}{3 + \cfrac{1}{8 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}}}}}$$

2. A similar mode of proceeding may also be used for converting any given surd, or tansendental quantity a, into a continued fraction, which is as follows:

Find such an integral value, α, of the given quantity, that $a - \alpha$ shall be less than unity; then $\frac{1}{a-\alpha}$ will be evidently greater than unity; which let, therefore, $= b$.

Find, also, such an integral value, β, of b, that $b - \beta$ shall be less than unity; then, as before, $\frac{1}{b-\beta}$ will be greater than unity; which let $= c$.

Proceed in the same manner with c, by putting its greatest integral value $= \gamma$; and thus, by continually following this process, it is evident that the value of a will be gradually exhausted, in as ready and simple a way as can be done, since none but integral numbers are employed in the operation; each of which approximates as nearly as possible to the value sought.

Hence it follows, that since $\frac{1}{a-\alpha}=b$, we shall have $a-\alpha=\frac{1}{b}$, or $a=\alpha+\frac{1}{b}$; and because $\frac{1}{b-\beta}=c$, we have likewise, $b-\beta=\frac{1}{c}$, or $b=\beta+\frac{1}{c}$, &c.; so that by the successive substitution of these values there will arise

$$a=\alpha+\frac{1}{b}=\alpha+\cfrac{1}{\beta+\cfrac{1}{c}}=a+\cfrac{1}{\beta+\cfrac{1}{\gamma+\cfrac{1}{d}\,\&c.}}$$

Or, by extending the number of terms, and generalizing the expression,

$$a=\alpha+\cfrac{1}{\beta+\cfrac{1}{\gamma+\cfrac{1}{\delta+\cfrac{1}{\varepsilon}\,\&c.}}}$$

Which series, in the present case, may be carried on indefinitely, the quantity a being supposed to be irrational, or of no exact determinate value.

3. On the other hand, any continued fraction may be readily converted into its equivalent common fraction, by first finding the value of a few of its leading terms, as in the reduction of complex fractions to simple ones, and then observing the law according to which they are derived.

Thus, if it were required to reduce the following continued fraction

$$x=\alpha+\cfrac{1}{\beta+\cfrac{1}{\gamma+\cfrac{1}{\delta}\,\&c.}}$$

to its corresponding vulgar fraction, the operation will stand as below:

$$\alpha=\frac{\alpha}{1},\ \alpha+\frac{1}{\beta}=\frac{\alpha\beta+1}{\beta},$$

$$\alpha+\cfrac{1}{\beta+\cfrac{1}{\gamma}}=\alpha+\frac{\gamma}{\beta\gamma+1}=\frac{\alpha\beta\gamma+\alpha+\gamma}{\beta\gamma+1}=\frac{(\alpha\beta+1)\gamma+\alpha}{\beta\gamma+1},$$

$$\alpha+\cfrac{1}{\beta+\cfrac{1}{\gamma+\cfrac{1}{\delta}}}=\alpha+\cfrac{1}{\beta+\cfrac{\delta}{\gamma\delta+1}}=\alpha+\frac{\gamma\delta+1}{\beta\gamma\delta+\beta+\delta}=$$

$$\frac{\alpha\beta\gamma\delta+\alpha\beta+\alpha\delta+\gamma\delta+1}{\beta\gamma\delta+\beta+\delta}=\frac{[(\alpha\beta+1)\gamma+\alpha]\delta+\alpha\beta+1}{(\beta\gamma+1)\delta+\beta}\ \&c.$$

Where it appears, that the successive sums of the continued fraction thus obtained, which are called *converging fractions,* are derived from each other by a regular process, which may be easily converted into a general rule for finding the sum of any number of terms whatever.

For this purpose, place the quotients α, β, γ, δ, ε, &c. in a right line, and the several results under them, as follows: beginning with $\frac{1}{0}$, in order to render the law of the terms more evident.

$$\begin{array}{cccccc} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \frac{1}{0}, & \frac{\alpha}{1}, & \frac{\alpha\beta+1}{\beta}, & \frac{\alpha\beta\gamma+\gamma+\alpha}{\beta\gamma+1}, & \frac{\alpha\beta\gamma\delta+\gamma\delta+\alpha\delta+\alpha\beta+1}{\beta\gamma\delta+\delta+\beta}, & \&c. \end{array}$$

Then the product of each numerator, and the quotient standing over it, being added to the preceding numerator, will give the numerator next following; and the product of each denominator, and its corresponding quotient, added to the preceding denominator, will give the following denominator.

As, for instance, if it be required to transform the continued fraction,

$$\cfrac{1}{7+\cfrac{1}{6+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{3}}}}}$$

into a series of converging fractions, we shall have

Denominators . . 0, 7, 6, 5, 2, 3,

Converging fractions $\frac{1}{0}$, $\frac{0}{1}$, $\frac{1}{7}$, $\frac{6}{43}$, $\frac{31}{222}$, $\frac{68}{487}$, $\frac{235}{1683}$

the last of which is the original fraction from which the continued fraction is derived.

4. Or, if A, B, C, &c., be made to represent the numerators of these fractions, taken in succession from $\frac{a}{1}$, and A′, B′, C′, &c., their denominators, we shall have, by the same rule,

$$\begin{array}{l|l} A = \alpha & A' = 1 \\ B = A\beta + 1 & B' = A'\beta \\ C = B\gamma + A & C' = B'\gamma + A' \\ D = C\delta + B & D' = C'\delta + B' \\ \quad\&c. & \quad\&c. \end{array}$$

And, consequently,

$$\frac{A}{A'} = \frac{\alpha}{1},\ \frac{B}{B'} = \frac{A\beta + 1}{A'\beta},\ \frac{C}{C'} = \frac{B\gamma + A}{B'\gamma + A'},\ \frac{D}{D'} = \frac{C\delta + B}{C'\delta + B'}, \&c.$$

So that if $\frac{P^{\circ}}{Q^{\circ}}$ and $\frac{P}{Q}$ be any two consecutive fractions, and μ a new quotient, or the last of those which enter into the next following fraction $\frac{P'}{Q'}$, we shall have, according to the above law of the terms, $\frac{P'}{Q'} = \frac{P\mu + P^{\circ}}{Q\mu + Q^{\circ}}$; from which it is evident, that the value of any fraction in the series can always be determined from the values of the two fractions that immediately precede it.

5. If the quantity x, which is to be converted into a continued fraction, be a rational fraction, it is evident, that this will be the last that occurs in the above series, as the continued fraction will terminate at that point; but if x be irrational, the continued fraction, into which it is developed, may be prolonged indefinitely.

6. Also, since the several quotients, α, β, γ, δ, &c. are here supposed to be all positive, the series of successive fractions, $\frac{1}{0}$, $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. which are called *principal fractions*, will be alternately greater and less than the total value x, of the continued fraction; which must, therefore, always necessarily fall between any one of these fractions and that next following.

7. It may likewise be shown, that the fractions, $\frac{1}{0}$, $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. thus obtained, converge nearer and nearer to the real value of x the farther they are continued; and that each of them expresses that value more accurately than any other fraction whatever, having a less denominator, or that can be conceived in more simple terms.

8. It may also be farther proved, that, if the terms of any two consecutive fractions $\frac{P^{\circ}}{Q^{\circ}}$, $\frac{P}{Q}$, in this series, be multiplied crosswise, the difference of the products $PQ^{\circ} - P^{\circ}Q$ thus arising, will be $= +1$ or -1, according as the fraction $\frac{P}{Q}$ is greater or less than x; and that the error committed by

taking any fraction in the series for the value of the total fraction, is always less than unity divided by the square of the denominator of that fraction.

9. To this we may add, that, since the principal converging fractions $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. are alternately less and greater than x, or the total value of the continued fraction, they may be separated into the two following classes.

$$\frac{A}{A'},\ \frac{C}{C'},\ \frac{E}{E'},\ \frac{G}{G'}\ \&c.$$

$$\frac{B}{B'},\ \frac{D}{D'},\ \frac{F}{F'},\ \frac{H}{H'}\ \&c.$$

In the first of which they are all less than x, or the total value of the continued fraction, but approaching nearer to it at every term; and in the second they are all greater than x, but each of them coming nearer to it, the farther they are continued.

And since, in the upper class of these fractions, it may be shown that

$$\frac{C}{C'}-\frac{A}{A'}=\frac{\gamma}{A'C'},\ \frac{E}{E'}-\frac{C}{C'}=\frac{\varepsilon}{E'C'}\ \&c.$$

and in the under, or second class, that

$$\frac{B}{B'}-\frac{D}{D'}=\frac{\delta}{B'D'},\ \frac{D}{D'}-\frac{F}{F'}=\frac{\zeta}{D'F'}\ \&c.$$

it follows, from what has been said above, that if γ, δ, ε, &c. were all equal to unity, it would be impossible to insert between any two consecutive fractions, in either of these series, any other fraction, having a less denominator than the greatest denominator of these fractions.

10. But, on the contrary, if the numbers γ, δ, ε, &c. be greater than unity, there may be inserted be-

tween the fractions in question, as many other *intermediate* or *secondary* fractions, as there are units in the numbers $\gamma - 1$, $\delta - 1$, $\varepsilon - 1$, &c.

Whence, if these fractions be so interpolated, we shall have the two following series; the several terms of each of which express the value of x more nearly, than it can be represented by any other fraction having a less denominator.

Increasing fractions less than x.

$$\frac{A}{A'},\ \frac{B+A}{B'+A'},\ \frac{2B+A}{2B'+A'},\ \frac{3B+A}{3B'+A'}\ \&c.\ \frac{\gamma B+A}{\gamma B'+A'}$$

$$\frac{C}{C'},\ \frac{D+C}{D'+C'},\ \frac{2D+C}{2D'+C'},\ \frac{3D+C}{3D'+C'}\ \&c.\ \frac{\varepsilon D+C}{\varepsilon D'+C'}$$

$$\frac{E}{E'},\ \frac{F+E}{F'+E'},\ \frac{2F+E}{2F'+E'},\ \frac{3F+E}{3F'+E'},\ \&c.\ \frac{\eta F+E}{\eta F'+E'}$$

&c.

Decreasing fractions greater than x.

$$\frac{A+1}{1},\ \frac{2A+1}{2},\ \frac{3A+1}{3},\ \frac{4A+1}{4},\ \&c.\ \frac{\beta A+1}{B}$$

$$\frac{B}{B'},\ \frac{C+B}{C'+B'},\ \frac{2C+B}{2C'+B'},\ \frac{3C+B}{3C'+B'},\ \&c.\ \frac{\delta C+B}{\delta C'+B'}$$

$$\frac{D}{D'},\ \frac{E+D}{E'+D'},\ \frac{2E+D}{2E'+D'},\ \frac{3E+D}{3E'+D'},\ \&c.\ \frac{\zeta E+D}{\zeta E'+D'}$$

&c.

Where it is to be observed, that, when the quantity x is irrational, or transcendental, neither of the two preceding series will ever terminate, because the series of principal fractions, $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$ &c. goes on, in this case, to infinity; but if x be rational, the first of the two series will terminate, but the second will proceed indefinitely.

As an example of one of the most useful cases, in the method here treated of, let it be proposed to express, by a continued fraction, the ratio of the circumference of the circle to the diameter; which, by stopping at the fifth place of decimals, is known to be $\frac{3.14159}{1.00000}$.

Here, by following the rule pointed out in the first part of this article, we shall have

```
100000)314159(3
       300000
       ------
        14159)100000(7
               99113
               -----
                 887)14159(15
                      887
                      ----
                      5289
                      4435
                      ----
                       854)887(1
                           854
                           ---
                            33 &c.
```

And consequently $\frac{3.14159}{1.00000} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{25 \,\&c.}}}}$

Whence, since $\alpha = 3$, $\beta = 7$, $\gamma = 15$, $\delta = 1$, &c. the several results, or approximate values of the continued fraction, thus arising, will be obtained as follows.

$A = \alpha = 3$	$A' = 1$
$B = A\beta + 1 = 3 \times 7 + 1 = 22$	$B' = A'\beta = 1 \times 7 = 7$
$C = B\gamma + A = 22 \times 15 + 3 = 333$	$C' = B'\gamma + A' = 7 \times 15 + 1 = 106$
$D = C\delta + B = 333 \times 1 + 22 = 355$	$D' = C'\delta + B' = 106 \times 1 + 7 = 113$
&c.	&c.

And therefore $\frac{A}{A'}=\frac{3}{1}$, $\frac{B}{B'}=\frac{22}{7}$, $\frac{C}{C'}=\frac{333}{106}$, $\frac{D}{D'}=\frac{355}{113}$ &c. are the *principal fractions* required; which are alternately less and greater than the circumference divided by the diameter, but each of them expressed in the least numbers possible; the second $\frac{22}{7}$, being the proportion assigned by Archimedes, and the fourth, $\frac{355}{113}$, that given by Metius, which is much more accurate than the former.

Also, since $\gamma=15$, if we take the fractions $\frac{3}{1}$ and $\frac{333}{106}$, which are each less than the truth, we can insert fourteen other *intermediate* or *secondary fractions* between them, as follows:

$\frac{A}{A'}=\frac{3}{1}$	$\frac{A+1}{1}=\frac{3+1}{1}=\frac{4}{1}$
$\frac{B+A}{B'+A'}=\frac{22+3}{7+1}=\frac{25}{8}$	$\frac{2A+1}{2}=\frac{6+1}{2}=\frac{7}{2}$
$\frac{2B+A}{2B'+A'}=\frac{44+3}{14+1}=\frac{47}{15}$	$\frac{3A+1}{3}=\frac{9+1}{3}=\frac{10}{3}$
$\frac{3B+A}{3B'+A'}=\frac{66+3}{21+1}=\frac{69}{22}$	$\frac{4A+1}{4}=\frac{12+1}{4}=\frac{13}{4}$
$\frac{4B+A}{4B'+A'}=\frac{88+3}{28+1}=\frac{91}{29}$	$\frac{5A+1}{5}=\frac{15+1}{5}=\frac{16}{5}$
&c.	&c.

Where $\frac{3}{1}$, $\frac{25}{8}$, $\frac{47}{15}$, $\frac{69}{22}$, $\frac{91}{29}$ &c. are all fractions less than the circumference divided by the diameter, and $\frac{4}{1}$, $\frac{7}{2}$, $\frac{10}{3}$, $\frac{13}{4}$, $\frac{16}{5}$ &c. are fractions greater than the circumference divided by the diameter; but each of them nearer the truth than any other fraction whatever, that can be expressed in more simple terms (*l*).

11. It may here be farther remarked, that, among continued fractions, those have been particularly distinguished, in which the denominators, after a certain number of changes, are continually repeated in the same order; in which case, the continued fraction, so formed, is said to be *periodical*, and may then always be considered as the root of a quadratic equation.

Thus, let $$x = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p \;\&c.}}}}$$

Then, since the number of these fractions is unlimited, and all of them, after the first, return again in the same order, it follows that their sum is also $= x$.

Whence $x = \frac{1}{p + x}$, or $x^2 + px = 1$; and consequently $x = -\frac{1}{2}p + \frac{1}{2}\sqrt{(p^2 + 4)}$.

In which case, the above continued fraction serves to determine the square root of the number $\sqrt{(p^2 + 4)}$; since we have

$$\tfrac{1}{2}\sqrt{(p^2 + 4)} = \tfrac{1}{2}p + x = \frac{p}{2} + \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p \;\&c.}}}$$

(*l*) By taking a greater number of decimals in the expression representing the circumference of the circle, the continued fraction, as well as the principal and intermediate fractions, may be carried to any extent required; but they will necessarily become more complex, the farther they are continued. Those who wish to see them at full length, may consult the *Oper. Math.* of Wallis, c. 2, tom. ii.

And if p, in this expression, be put $=2$, we shall have

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2\,\&c.}}}$$

A formula, which has the advantage of showing the law, by which the operation may be continued, more evidently than if the extraction had been performed in the usual way (*m*).

Again, let us take the following continued fraction,

$$x=\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{p+\cfrac{1}{q\,\&c.}}}}$$

in which the denominators recur periodically in pairs:

Then we shall have

$$x=\cfrac{1}{p+\cfrac{1}{q+x}}$$

from which there results the quadratic equation $px^2+pqx=q$; where

$$x+\frac{q}{2}=\frac{1}{2p}\sqrt{(p^2q^2+4pq)}$$

And if p, in this expression, be put $=2$, and $q=3$, we shall have

$$\sqrt{15}=3+2\left\{\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3\,\&c.}}}}\right\}$$

(*m*) From the common method of extracting the roots of numbers, we know that $\sqrt{2}=1.41421356$; but the bare inspection of this decimal leads to no rule for its farther extension; whereas, when expressed by a continued fraction, we see in what way it may be carried on to any degree of accuracy.

Where the law of continuation is equally obvious as in the former example; and, by substituting other numbers for p and q, various series of this kind may be obtained.

12. Also, if a continued fraction be irregular in some of its first terms, or only becomes periodic at a certain distance from its commencement, it may still be resolved in a similar manner with the former. Thus let

$$x=p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s+\cfrac{1}{r+\cfrac{1}{s\ \&c.}}}}}$$

Then, by making y equal to that part of the fraction which is periodic, we shall have

$$x=p+\cfrac{1}{q+\cfrac{1}{y}} \qquad \text{Or } y=\frac{x-p}{1+q(x-p)}$$

But $y=r+\cfrac{1}{s+\cfrac{1}{y}}$; whence, also, $sy^2-rsy=r$; and consequently, by substitution, and dividing each of the terms by s, there will arise the following quadratic equation:

$$\left\{\frac{x-p}{1+q(x-p)}\right\}^2-r\left\{\frac{x-p}{1+q(x-p)}\right\}=\frac{r}{s}$$

From which the value of $x-p$, or x, may be readily obtained, as below:

$$x=p+\frac{\frac{1}{2}r+\surd(\frac{r^2}{4}+\frac{r}{s})}{1-q\{\frac{1}{2}r+\surd(\frac{q^2}{4}+\frac{r}{s})\}}$$

In like manner, if there be taken the following continued fraction,

$$x=a+\cfrac{p}{q+\cfrac{p}{q+\cfrac{p}{q+\cfrac{p}{q\,\&\text{c.}}}}} \qquad \text{Or } x-a=\cfrac{p}{q+\cfrac{p}{q+\cfrac{p}{q\,\&\text{c.}}}}$$

we shall obtain, by a substitution similar to the former,

$$x-a=\frac{p}{q+x-a}, \text{ or } x=\frac{2a-q+\sqrt{(q^2+4p)}}{2}.$$

And consequently, by transposing $\frac{2a}{2}$, or a, to the other side of the equation,

$$\frac{\sqrt{(q^2+4p)}-q}{2}=\cfrac{p}{q+\cfrac{p}{q+\cfrac{p}{q\,\&\text{c.}}}}$$

Or, by making $q=2a$, the expression for the simple radical will become

$$\sqrt{a^2+p}=a+\cfrac{p}{2a+\cfrac{p}{2a+\cfrac{p}{2a\,\&\text{c.}}}}$$

And in nearly the same way may any other expression of this kind be transformed to a quadratic, or a surd; to which they are always reducible, whether the periodic part consists of one, two, or more terms, or whether it commences in a regular or irregular manner (n).

13. Continued fractions are also peculiarly ap-

(n) It has been long known, that any given continued periodic fraction can be reduced to a quadratic equation, and hence to a simple surd; but Lagrange appears to have been the first who has proved the reverse of this proposition, by showing that the square root of any whole number can always be expressed by a continued fraction. — See his work entitled *De la Résolution des Equation Numériques*, page 65.

plicable to the solution of unlimited equations, both of the first and other higher degrees; but before we apply them to that purpose, it will be proper to observe, that if one solution of any equation, of the form

$$ax - by = \pm c,$$

be known, all the rest may be readily determined from it, by the successive addition of b, or a, to the value last found.

For supposing that p and q are values of x and y, that will satisfy the conditions of the question, it can then be shown that

$$x = p = p + b = p + 2b = p + 3b \ . \ . \ . \ p + mb,$$
$$y = q = q + a = q + 2a = q + 3a \ . \ . \ . \ q + ma,$$

where m may be any positive or negative integer whatever; the number of answers, in this case, being obviously indefinite.

14. In like manner, if p and q be values of x and y, that satisfy the conditions of the equation

$$ax + by = c,$$

it may be readily shown, in the same manner as before, that

$$x = p = p + b = p + 2b = p + 3b \ . \ . \ . \ p + mb,$$
$$y = q = q - a = q - 2a = q - 3a \ . \ . \ . \ q - ma,$$

where m must be now so taken, that y shall be positive; the answers being here plainly limited to a certain number only, depending on the value of q.

Hence it appears, that if the least values of x and y, in the above equations, were known, all the rest of the values might be found from the series here given.

15. This having been premised, we shall now be able to deduce the general solution of indeterminate equations of the first degree, from the known properties of continued fractions, by a process similar to those before used.

For this purpose, let there be taken the first of the two former equations,

$$ax-by=\pm c,$$

where a and b are supposed to be prime to each other, as in Art. F, and make $x=pc$ and $y=qc$; which will give, by substitution,

$$ap-bq=\pm 1.$$

Then, if $\frac{b}{a}$ be converted into a continued fraction, by the method given in the first part of the present article, the least values of p and q, in this last equation, will be the numerator and denominator of that, among the principal fractions, deduced from them, which immediately precedes the last.

And consequently, from the values of p and q, thus found, we shall have

$$x=\pm pc \text{ and } y=\pm qc.$$

Or more generally, according to what has been before shown,

$$x=\pm pc+mb,\ y=\pm qc+ma.$$

Observing that the upper sign takes place when the principal fraction above mentioned is the second, fourth, sixth, &c. of the series; and the under sign when it is the first, third, fifth, &c.; beginning with the first fraction that arises, without reckoning $\frac{1}{0}$.

16. And if the equation that is to be resolved be of the form

$$ax + by = c,$$

we shall now have, by making $x = pc$ and $y = -qc$, and following the same rule as above,

$$x = \pm pc + mb, \quad y = \mp qc - ma,$$

where the upper and under sign must be taken in the same way as before.

As an example of this method, let it be required to find the several values of x and y in the indeterminate equation

$$39x - 56y = 11.$$

Here we have $a = 39$, $b = 56$, and $c = 11$.

And the fraction $\frac{56}{39}$, when reduced to a continued fraction, gives the following quotients and results, or principal fractions:

$$1, \quad 2, \quad 3, \quad 2, \quad 2.$$

$$\frac{1}{0}, \quad \frac{1}{1}, \quad \frac{3}{2}, \quad \frac{10}{7}, \quad \frac{23}{16}, \quad \frac{56}{39}.$$

Whence, the penultimate fraction being $\frac{23}{16}$, we have $p = 23$ and $q = 16$.

And as this fraction is the fourth in the series, it follows, from the rule given above, that

$$x = 23 \times 11 + 56m, \quad y = 16 \times 11 + 39m,$$

Where, if m be taken = any of the natural numbers ± 1, ± 2, ± 3, ± 4, &c. the several solutions of the question may be readily deduced.

Again, let $5x + 8y + 7z = 50$ be the equation to be resolved, or, which is the same,

$$5x + 8y = 50 - 7z.$$

Here we have $a = 5$, $b = 8$, and $c = 50 - 7z$.

And the fraction $\frac{8}{5}$, when reduced to a continued fraction, gives the following quotients and principal fractions:

$$1,\quad 1,\quad 1,\quad 2.$$

$$\frac{1}{0},\quad \frac{1}{1},\quad \frac{2}{1},\quad \frac{3}{2},\quad \frac{8}{5}.$$

Whence, the penultimate fraction $\frac{3}{2}$ being here the third in the series, we shall have

$$p=-3 \text{ and } q=-2.$$

And consequently, by following the rule for this case,

$$x=-3\times 50+3\times 7z+8m=21z-150+8m,$$
$$y=2\times 50-2\times 7z-5m=100-14z-5m.$$

In which expressions any numbers whatever may be taken for m and z, that will render the values of x and y positive.

These examples will be sufficient to show the method of applying the doctrine of continued fractions to the solution of indeterminate problems of the first degree, of whatever form they may be.

16. But before we proceed to those of the second degree, it will be proper to observe, that the method more generally employed for converting any rational or surd quantity x into a continued fraction is by successively substituting

$$x=a+\frac{1}{x'},\quad x'=a'+\frac{1}{x''},\quad x''=a''+\frac{1}{x'''},\ \text{\&c.}$$

a being the greatest integral number contained in x, a' the greatest integer contained in x', &c.

So that, by proceeding in this manner, it is evident that x will be transformed into the continued fraction

$$a+\cfrac{1}{a'+\cfrac{1}{a''+\cfrac{1}{a'''\,\&c.}}}$$

which will consist of a finite, or an infinite, number of terms, according as the quantity x is rational or irrational.

In which case it is to be remarked, that the successive terms a, a', a'', &c. of the continued fraction, are the *partial quotients*, before mentioned; and the quantities x, x', x'', &c. resulting from the development of x, and of which the integers a, a', a'', &c. form the greater part, are the *complete quotients*.

Hence, every complete quotient, in this mode of conversion, comprehends, implicitly, besides the integer which is contained in it, all the following quotients of the continued fraction; since it readily appears, that by the development of this quotient we can obtain all the rest.

And consequently, if we have any algebraical expression, which represents the value of the continued fraction, prolonged to the term $a^{(n)}$ inclusively, and there be substituted in it the complete quotient $x^{(n)}$ instead of $a^{(n)}$, it is plain that the result will be the exact value of x, even in the case where the continued fraction is supposed to be carried on to infinity; for

$$x=a+\frac{1}{x'},\quad x=a+\cfrac{1}{a'+\cfrac{1}{x''}},\quad x=a+\cfrac{1}{a'+\cfrac{1}{a''+\cfrac{1}{x'''\,\&c.}}}$$

17. Whence it follows, that by means of each of these complete quotients, we can always readily

obtain the entire and exact value of the quantity proposed, however far the development may have been continued, which circumstance will be found of considerable utility in extracting the square roots of numbers.

Thus, if it were required, as an example of this kind, to extract the square root of 19, by means of a continued fraction, it may be done by continually finding the largest integer number contained in the several values of x, x', x'', &c. as follows:

$$x=\sqrt{19}=4+\frac{\sqrt{19}-4}{1}$$

$$x'=\frac{1}{\sqrt{19}-4}=\frac{\sqrt{19}+4}{3}=2+\frac{\sqrt{19}-2}{3}$$

$$x''=\frac{3}{\sqrt{19}-2}=\frac{\sqrt{19}+2}{5}=1+\frac{\sqrt{19}-3}{5}$$

$$x'''=\frac{5}{\sqrt{19}-3}=\frac{\sqrt{19}+3}{2}=3+\frac{\sqrt{19}-3}{2}$$

$$x^{\text{iv}}=\frac{2}{\sqrt{19}-3}=\frac{\sqrt{19}+3}{5}=1+\frac{\sqrt{19}-2}{5}$$

$$x^{\text{v}}=\frac{5}{\sqrt{19}-2}=\frac{\sqrt{19}+2}{3}=2+\frac{\sqrt{19}-4}{3}$$

$$x^{\text{vi}}=\frac{3}{\sqrt{19}-4}=\frac{\sqrt{19}+4}{1}=8+\frac{\sqrt{19}-4}{1}$$

$$x^{\text{vii}}=\frac{1}{\sqrt{19}-4}=\frac{\sqrt{19}+4}{3}=2+\frac{\sqrt{19}-2}{3}$$

Where, since x^{vii} is $=x'$, it is plain that the quotients 2, 1, 3, 1, 2, 8, already found, will always return again, in the same order, to infinity.

Whence

$$\sqrt{19}=4+\frac{1}{2+\frac{1}{1+\frac{1}{3+\frac{1}{1+\frac{1}{2+\frac{1}{8\&c.}}}}}}$$

And consequently, by putting $\alpha=4$, $\beta=2$, $\gamma=1$, $\delta=3$, $\varepsilon=1$, $\zeta=2$, $\eta=8$, we shall have, according to the method before described.

$A=4$	$A'=1$
$B=A\beta+1=4\times2+1=9$	$B'=A'\beta=1\times2=2$
$C=B\gamma+A=9\times1+4=13$	$C'=B'\gamma+A'=2\times1+1=3$
$D=C\delta+B=13\times3+9=48$	$D'=C'\delta+B'=3\times3+2=11$
$E=D\varepsilon+C=48\times1+13=61$	$E'=D'\varepsilon+C'=11\times1+3=14$
$F=E\zeta+D=61\times2+48=170$	$F'=E'\zeta+D'=14\times2+11=39$
$G=F\eta+E=170\times8+61=1421$	$G'=F'\eta+E'=39\times8+14=326$
&c.	&c.

Whence, the principal converging fractions approximating to the $\surd 19$, are

$$\frac{4}{1},\ \frac{9}{2},\ \frac{13}{3},\ \frac{48}{11},\ \frac{61}{14},\ \frac{170}{39},\ \frac{1421}{326}\ \&c.$$

the last of which, when converted into decimals, will be found correct to six places of figures.

And in the same way may the square root of any other nonquadrate number be reduced to a continued fraction; and thence its approximate value determined as above.

18. This being premised, it may now be observed, that the least integral values of x and y, in any indeterminate equation of the form

$$x^2-Ay^2=\pm1,$$

in which A is not a square, can always be found, by converting $\surd A$ into a continued fraction, and then calculating the results, or principal fractions, answering to the several quotients, to the end of the first period, as in the instance above given.

In which case, the numerator and denominator of the penultimate fraction, so determined, will give the respective values of x and y in the equation

$$x^2-Ay^2=1,\ \text{or}\ x^2-Ay^2=-1,$$

according as the period consists of an even or an odd number of quotients.

Thus, if it were required to find the least values of x and y in the equation

$$x^2-19y^2=1,$$

we shall have, for $\sqrt{19}$, according to the example just given, the following quotients and converging fractions, to the end of the first period.

Quotients $4,\ 2,\ 1,\ 3,\ 1,\ 2,\ 8$

Fractions $\frac{1}{0},\ \frac{4}{1},\ \frac{9}{2},\ \frac{13}{3},\ \frac{48}{11},\ \frac{61}{14},\ \frac{170}{39},\ \frac{1421}{326}$

Whence, the number of quotients, 2, 1, 3, 1, 2, 8, in the recurring period, being even, and $\frac{170}{39}$ the penultimate fraction in that period, it follows, that $x=170$ and $y=39$; which numbers will be found to answer the conditions of the question.

Again, let there be given $x^2-13y^2=-1$, to find the least values of x and y in whole numbers.

Here, if $\sqrt{13}$ be converted into a continued fraction, as before, we shall have the following quotients and fractions, to the end of the first period.

Quotients $3,\ 1,\ 1,\ 1,\ 1,\ 6$

Fractions $\frac{1}{0},\ \frac{3}{1},\ \frac{4}{1},\ \frac{7}{2},\ \frac{11}{3},\ \frac{18}{5},\ \frac{119}{33}$

And since the number of quotients in the recurring period, 1, 1, 1, 1, 6, is, in this case, odd, and $\frac{18}{5}$ is the penultimate fraction, it follows, that $x=18$ and $y=5$.

19. Hence, also, because the periodical part of the quotients, both in this and the former continued fraction, must return again, in the same

order, to infinity, it is plain, that questions of this kind admit of an indefinite number of solutions; since all the penultimate fractions, in each of these several periods, will equally answer the conditions of the question.

20. To this we may also farther add, that if $\frac{p}{q}$ be made to represent the penultimate fraction that resolves the equation $x^2-\text{A}y^2=-1$, in which the absolute term is negative, $\frac{2\text{A}q^2-1}{2pq}$ will be the fraction that resolves the equation $x^2-\text{A}y^2=1$, in which the absolute term is positive.

Thus, $\frac{p}{q}=\frac{18}{5}$ being found, as above, to be the fraction which resolves the equation $x^2-13y^2=-1$,

$$\frac{2\text{A}q^2-1}{2pq}=\frac{2\times13\times5^2-1}{2\times18\times5}=\frac{650-1}{180}=\frac{649}{180},$$

will be the fraction that resolves the equation $x^2-13y^2=1$; x being $=649$, and $y=180$ (*o*).

21. So that, although the equation $x^2-\text{A}y^2=-1$, can only be resolved in integers when the number of quotients in the recurring period of the continued fraction, to which $\sqrt{\text{A}}$ is reduced, is odd, the

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(*o*) In many examples of this kind, the least numbers that will satisfy the conditions of the equation $x^2-\text{A}y^2=1$, are often very considerable.

Thus, the most simple solution of the equation $x^2-211y^2=1$, is

$x=278354373650$, and $y=19162705353$.

And the most simple solution of the equation $x^2-991y^2=1$, is

$x=379516400906811930638014896080$, and

$y=12055735790331359447442538767$.

From which it appears how necessary it is to have an infallible method of proceeding in these cases; as we should be, equation $x^2 - Ay^2 = 1$, on the contrary, always admits of an indefinite number of such answers, provided A be not a square; because, in that case, the difference of two integral squares would be $= 1$, which is impossible.

22. In like manner, if there be given any indeterminate equation of the more general form

$$x^2 - Ay^2 = \pm B,$$

in which B is less than $\sqrt{A}$, the fraction $\frac{x}{y}$, when the equation is resolvable, will be always comprised among the principal fractions, in the first period, arising from the conversion of $\sqrt{A}$ into a continued fraction.

As, for instance, if the fraction $\frac{x}{y}$, found as above, be of an even rank in this period, or less than $\sqrt{A}$, it will give the solution of the equation $x^2 - Ay^2 = B$; and if it be of an odd rank, or greater than $\sqrt{A}$, it will give the solution of the equation $x^2 - Ay^2 = -B$; and the same will be true if there should be two or more fractions of this kind in the period.

23. But if, after trying all the converging frac-

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greatly deceived, if, after having failed in trying a few moderately large numbers, we should thence conclude that the question was impossible.

With respect to the other answers of which these questions are susceptible, if m and n be made to represent the least values of x and y, as above found, the rest of the solutions, in the case $x^2 - Ay^2 = 1$, may be readily derived from the equation $x + y\sqrt{A} = (m + n\sqrt{A})^k$, and those of the case $x^2 - Ay^2 = -1$, from the equation $x + y\sqrt{A} = (m + n\sqrt{A})^{2k+1}$; where k, being taken $= 1, 2, 3$, &c. successively, the comparison of the like terms of the expression will give the several values of x and y.

tions, answering to the several quotients in the first period, no one should be found that satisfies the conditions of the question, it may be safely concluded, that neither the equation $x^2 - \mathrm{A}y^2 = +\mathrm{B}$, nor $x^2 - \mathrm{A}y^2 = -\mathrm{B}$, can be resolved in whole numbers.

Thus, as an example of this method, let it be required to find the least integral values of x and y in the equation

$$x^2 - 19y^2 = -3.$$

Here, since $\sqrt{19}$, when converted into a continued fraction, has been shown to give, in its first period, the following quotients and converging fractions.

$$\begin{array}{cccccccc} 4, & 2, & 1, & 3, & 1, & 2, & 8 & \\ \frac{1}{0}, & \frac{4}{1}, & \frac{9}{2}, & \frac{13}{3}, & \frac{48}{11}, & \frac{61}{14}, & \frac{170}{39}, & \frac{1421}{326} \end{array}$$

we shall find, by trial, that both $\frac{61}{14}$ and $\frac{1421}{326}$, which are the fifth and seventh terms of the period, will satisfy the conditions of the question.

For, if 61 be substituted for x, and 14 for y, we shall have $x^2 - 19y^2 = 61^2 - 19 \times 14^2 = 3721 - 3724 = -3$.

And, if 1421 be put for x, and 326 for y, we shall also have $x^2 - 19y^2 = 2019241 - 2019244 = -3$, as before.

Again, let it be required to find the least integral values of x and y in the equation

$$x^2 - 55y^2 = 5.$$

Here, by extracting $\sqrt{55}$, according to the method employed in a former example, there will arise

$$\sqrt{55}=7+\frac{\sqrt{55}-7}{1}$$

$$x'=\frac{1}{\sqrt{55}-7}=\frac{\sqrt{55}+7}{6}=2+\frac{\sqrt{55}-5}{6}$$

$$x''=\frac{6}{\sqrt{55}-5}=\frac{\sqrt{55}+5}{5}=2+\frac{\sqrt{55}-5}{5}$$

$$x'''=\frac{5}{\sqrt{55}-5}=\frac{\sqrt{55}+5}{6}=2+\frac{\sqrt{55}-7}{6}$$

$$x^{\text{iv}}=\frac{6}{\sqrt{55}-7}=\frac{\sqrt{55}+7}{1}=14+\frac{\sqrt{55}-7}{1}$$

&c. &c.

Where, the fractional part of the last term being equal to that of the first, we shall have, for the several quotients of the first period,

$$\begin{array}{ccccc} \alpha & \beta & \gamma & \delta & \varepsilon \\ 7, & 2, & 2, & 2, & 14 \end{array}$$

And, consequently, by following the second method, pointed out in the first part of this article, there will arise

$A=\alpha=7$	$A'=1$
$B=A\beta+1=7\times2+1=15$	$B'=A'\beta=1\times2=2$
$C=B\gamma+A=15\times2+7=37$	$C'=B'\gamma+A'=2\times2+1=5$
$D=C\delta+B=37\times2+15=89$	$D'=C'\delta+B'=5\times2+2=12$
$E=D\varepsilon+C=89\times14+37=1283$	$E'=D'\varepsilon+C'=12\times14+5=173$
&c.	&c.

Whence, the several converging fractions, corresponding with the terms of this period, will be

$$\frac{7}{1},\quad\frac{15}{2},\quad\frac{37}{5},\quad\frac{89}{12},\quad\frac{1283}{173}.$$

And by trial, it will be readily found, that the second of these fractions, $\frac{15}{2}$, will satisfy the conditions of the question.

For $x^2-55y^2=15^2-55\times2^2=225-220=5$, as was required.

And if it were proposed to find the least values of x and y in the equation

$$x^2 - 55y^2 = 1,$$

the penultimate fraction, $\frac{89}{12}$, in the above series, will be found to satisfy the conditions of the question.

For $x^2 - 55y^2 = 89^2 - 55 \times 12^2 = 7921 - 7920 = 1$, as was to be shown (*p*).

QUESTIONS FOR PRACTICE.

1. It is required to find the least fraction, having only two figures in each of its terms, that shall approach the nearest possible in value to $\frac{1103}{887}$.

Ans. $\frac{97}{78}$

2. It is required to find the ratios next less and next greater than $\frac{450}{109}$, that can be each expressed by fractions having only two figures in their denominators.

Ans. $\frac{289}{70}$ & $\frac{128}{31}$ (*q*)

(*p*) Here, as in the former part of this problem, if m and n be made to represent the several successive values of x and y in the equation $x^2 - \mathrm{A}y^2 = \pm 1$, and p and q be the least values of x and y in the equation $x^2 - \mathrm{A}y^2 = \pm \mathrm{B}$, as above found, we shall have, in general terms,

$$x = pm \pm \mathrm{A}qn, \text{ and } y = pn \mp qm,$$

from which the various solutions, that any question of this kind is susceptible of, may be readily determined.

(*q*) This question arises out of the method usually employed for adjusting the civil mode of reckoning time to that shown by the sun; the ratio $\frac{450}{109}$, being that of 24h. to 5h. 48m. 48 sec., the time by which the solar year exceeds that in common use;

3. Given $9x+13y=200$, to find all the values of x and y in whole positive numbers.

Ans. $x=19, 2$; $y=5, 14$.

4. Given $256x-87y=1$, to find the least possible values of x and y in whole numbers.

Ans. $x=52$, $y=153$

5. In how many different ways is it possible to pay 100*l.* with seven shilling pieces and dollars at 4*s*. 6*d.* each? Ans. 6 different ways.

6. Given $11x+14y+19z=960$, to find the three values of x, y, and z, in whole numbers, that approach the nearest to equality.

Ans. $x=23$, $y=22$, $z=21$

7. It is required to find the least values of x and y in the indeterminate equation $7x+19y=1921$.

Ans. $x=3$, $y=100$

8. It is required to find the number of solutions, in positive integers, of the equation $7x+9y+23z=9999$. Ans. 34365

9. Given $x^2-61y^2=-1$, to find the least values of x and y in whole positive numbers.

Ans. $x=29718$, and $y=3805$

10. Given $x^2-85y^2=1$, to find the least values of x and y in whole positive numbers.

Ans. $x=285769$, $y=30996$

11. Given $x^2-61y^2=-5$, to find the least values of x and y in whole positive numbers.

Ans. $x=164$, $y=21$

12. Given $x^2-61y^2=5$, to find the least values of x and y in whole positive numbers.

Ans. $x=453$, $y=58$

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and, when investigated at length, will show the proper number of intercalary days that answer to a given number of years.

OF THE DECOMPOSITION OF RATIONAL FRACTIONS INTO SIMPLE FRACTIONS.

(1) THIS problem being of considerable utility in some of the higher branches of analysis, but more particularly in the integral calculus, where it is of frequent occurrence, it will be here proper to show some of the most simple methods of effecting the decomposition required; which, in many of the most useful cases, may be done, without the assistance of fluxions, as follows:

RULE I.

1. Let the fraction $\frac{\text{N}}{\text{D}}$, to be decomposed, have, at least, one dimension less in its numerator than in the denominator; to which state it can be always reduced, when necessary, by division; and will then have the form

$$\frac{\text{N}}{\text{D}}=\frac{a+bx+cx^2+dx^3+\;\ldots\ldots\;px^{m-1}}{a'+b'x+c'x^2+d'x^3+\;\ldots\ldots\;q'x^m}.$$

2 Find all the m simple factors into which the denominator D can be reduced, either by trial, or by putting the sum of its terms $=0$, and then determining the roots r, r', r'', &c. of the equation, so formed, in the usual way; which being done, the original equation may now be represented under the form

$$\frac{\text{N}}{\text{D}}=\frac{a+bx+cx^2+dx^3+\;\ldots\ldots\;px^{m-1}}{(x-r)(x-r')(x-r'')\;\ldots\ldots\;[x-r^{(n-1)}]}.$$

3. Let A, A$'$, A$''$, be the numerators of the several simple fractions into which the given fraction is re-

solvable, and we shall have, in the case where the factors of the denominator are all unequal,

$$\frac{N}{D}=\frac{A}{x-r}+\frac{A'}{x-r'}+\frac{A''}{x-r''}+\;\ldots\ldots\;\frac{A^{(n-1)}}{x-r^{(n-1)}}.$$

Where the values of A, A′, A″. &c. may be found by reducing the fractions to a common denominator, (which will be the same as D,) and then comparing the coefficients of the like powers of x together, according to the method pointed out in a former article.

It is necessary to be observed, however, with regard to this rule, that, if the highest term of the denominator of the given fraction has any coefficient prefixed to it, the product of the factors, found by determining the roots of the above mentioned equation, must be multiplied by this coefficient, in order to obtain the true factors.

EXAMPLES.

1. It is required to convert the rational fraction $\frac{1+x}{x-x^2}$ into its equivalent simple fractions.

Here the factors of the denominator $x-x^2$ being x and $1-x$, we shall have

$$\frac{1+x}{x-x^2}=\frac{A}{x}+\frac{A'}{1-x};\text{ or }\frac{1+x}{x-x^2}=\frac{A+(A'-A)x}{x-x^2}.$$

And, by comparing the like terms of the two numerators, $A=1$ and $A'-A=1$, or $A'=1+A=1+1=2$; whence $\frac{1+x}{1-x^2}=\frac{1}{x}+\frac{2}{1-x}=$ the fractions required.

2. It is required to convert the rational fraction $\frac{1+x^2}{x-x^3}$ into its equivalent simple fractions.

Here, the factors of the denominator $x-x^3$ being x, $1-x$, and $1+x$, we shall have

$$\frac{1+x^2}{x-x^3}=\frac{A}{x}+\frac{A'}{1-x}+\frac{A''}{1+x};$$

or, by reducing the fractions to the same denominator,

$$\frac{1+x^2}{x-x^3}=\frac{A(1-x^2)+A'(x+x^2)+A''(x-x^2)}{x-x^3};$$

whence $1+x^2=A+(A'+A'')x+(-A+A'-A'')x^2$; and consequently, by equating the coefficients of the like terms, $A=1$, $A'=1$, and $A''=-1$; therefore $\frac{1+x^2}{x-x^3}=\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}$, are the simple fractions required.

3. It is required to convert the rational fraction $\frac{1-x}{1-4x+2x^2}$ into its equivalent simple fractions.

Here, by making the denominator $2x^2-4x+1=0$, the two roots of the equation will be found $=1\pm\frac{1}{2}\sqrt{2}$; and consequently the factors are $2(1+\frac{1}{2}\sqrt{2}-x)(1-\frac{1}{2}\sqrt{2}-x)$; whence

$$\frac{1-x}{1-4x+2x^2}=\frac{A}{(2+\sqrt{2})-2x}+\frac{A'}{(1-\frac{1}{2}\sqrt{2})-x};$$

or, by reduction,

$$\frac{1-x}{1-4x+2x^2}=\frac{(1-\frac{1}{2}\sqrt{2})A-Ax+(2+\sqrt{2})A'-2A'x}{1-4x+2x^2}=$$
$$\frac{A+2A'+(A'-\frac{1}{2}A)\sqrt{2}-(A+2A')x}{1-4x+2x^2}.$$

And putting $A+2A'+(A'-\frac{1}{2}A)\sqrt{2}=1$ and $A+2A'=1$, we shall have, from the resolution of these equations, $A=\frac{1}{2}$ and $A'=\frac{1}{4}$; therefore $\frac{1-x}{1-4x+2x^2}=\frac{\frac{1}{2}}{2+\sqrt{2}-2x}+\frac{\frac{1}{4}}{1-\frac{1}{2}\sqrt{2}-x}$, the fractions required.

Note. If two or more of the factors of the denominator of the given fraction be equal to each other, the decomposition of it into simple fractions cannot, in this case, take place in the way pointed out by the rule; but will then be of the following form (r):

$$\frac{\text{N}}{\text{D}}=\frac{\text{B}}{(x-p)^n}+\frac{\text{B}'}{(x-p)^{n-1}}\ \cdots\cdots\ \frac{\text{B}^{(n-1)}}{x-p}+\frac{\text{P}}{\text{S}};$$

where p represents the root of one of the equal factors contained in D, and $\frac{\text{P}}{\text{S}}$ the sum of the partial factors due to the unequal factors in D, of which A, A′, &c. are the numerators. Thus,

4. Let it be required to convert $\frac{1+x}{(1-x)^2}$ into its equivalent simple fractions.

Here, since the denominator $(1-x)^2$ is the product of $(1-x)(1-x)$, we shall have

$$\frac{1+x}{(1-x)^2}=\frac{\text{B}}{(1-x)^2}+\frac{\text{B}'}{1-x}=\frac{\text{B}(1-x)+\text{B}'(1-x)^2}{(1-x)(1-x)^2}=$$
$$\frac{\text{B}+\text{B}'(1-x)}{(1-x)^2}=\frac{\text{B}+\text{B}'-\text{B}'x}{(1-x)^2};$$

and, by equating the homologous terms, $\text{B}'=-1$ and $\text{B}+\text{B}'=1$, or $\text{B}=2$; whence $\frac{1+x}{(1-x)^2}=\frac{2}{(1-x)^2}-\frac{1}{1-x}$, the simple fractions required.

(r) In order to show that the former part of the rule is inapplicable in the case where there are equal factors in the denominator, let us suppose, for instance, that $\frac{a+bx}{(x-c)^2}=\frac{\text{A}}{x-c}+\frac{\text{A}'}{x-c}$; then we shall have, by reduction, $\frac{a+bx}{(x-c)^2}=\frac{(\text{A}+\text{A}')(x-c)}{(x-c)^2}=\frac{(\text{A}+\text{A}')x-c(\text{A}+\text{A}')}{(x-c)^2}$; and consequently, by comparing the homologous terms, $\text{A}+\text{A}'=b$, and $c(\text{A}+\text{A}')=-a$; which results are, evidently, inadmissible.

5. It is required to convert $\frac{1-5x}{(1-x)(1+x)^2}$ into its equivalent simple fractions.

Here, according to the above rule,

$$\frac{1-5x}{(1-x)(1+x)^2}=\frac{\text{B}}{(1+x)^2}+\frac{\text{B}'}{1+x}+\frac{\text{A}}{1-x}=$$
$$\frac{\text{B}[(1+x)(1-x)]+\text{B}'[(1+x)^2(1-x)]+\text{A}(1+x)^3}{(1-x)(1+x)^3}=$$
$$\frac{\text{B}(1-x)+\text{B}'(1-x^2)+\text{A}(1+2x+x^2)}{(1-x)(1+x)^2}=$$
$$\frac{\text{B}+\text{B}'+\text{A}+(2\text{A}-\text{B})x+(\text{A}-\text{B}')x^2}{(1-x)(1+x)^2};$$

whence, by putting $\text{B}+\text{B}'+\text{A}=1$, $2\text{A}-\text{B}=-5$, and $\text{A}-\text{B}'=0$, we shall have $\text{B}=3$, $\text{B}'=-1$, and $\text{A}=-1$; consequently $\frac{1-5x}{(1-x)(1+x)^2}=\frac{3}{(1+x)^2}-\frac{1}{1+x}-\frac{1}{1-x}$, are the simple fractions required.

It may here also be farther remarked, that, when the factors of the denominator of the given fraction are few in number, we can always readily determine the simple fractions into which it is convertible by the preceding methods; but as the calculation, in other cases, becomes more laborious, it will be necessary to show, that any one of the numerators of these fractions may be deduced immediately from N and D, independently of the rest, which may be done as follows:

RULE II.

1. When the factors of the denominator of the given fraction are all unequal, or of the form

$$\frac{\text{N}}{\text{D}}=\frac{\text{A}}{x-r}+\frac{\text{A}'}{x-r'}+\frac{\text{A}''}{1-r''}+\frac{\text{A}'''}{1-r'''}\ \&\text{c.}$$

take that which constitutes the denominator of the simple fraction that is to be found, and let S denote the product of all the remaining factors; then, if the root, or value of x in that factor, be substituted for x in the formula $\frac{N}{S}$, it will give the numerator of the fraction required.

2. If some of the factors are equal and others unequal, or of the form

$$\frac{N}{D}=\frac{A}{x-r}+\frac{A'}{x-r''}\&c. \; \ldots \; +\frac{B}{(x-p)^n}+\frac{B'}{(x-p)^{n-1}}\&c.$$

let S denote the product of all the factors in the denominator, except one, as before; then find the simple fractions due to the unequal factors by the first part of the rule; and, for the unequal factors, proceed as follows:

1. $B=\frac{N}{S}$, taking x in N and $S=p$.

Let $Q=\frac{N-BS}{x-p}$, then

2. $B'=\frac{Q}{S}$, taking x in Q, and $S=p$.

Let $Q'=\frac{Q-B'S}{x-p}$, then

3. $B''=\frac{Q'}{S}$, taking x in Q′, and $S=p$.

Let $Q''=\frac{Q'-B''S}{x-p}$, then

4. $B'''=\frac{Q''}{S}$, taking x in Q″, and $S=p$.

&c. &c.

Which operation being performed, the sum of the fractions, thus obtained, together with the former, will give all the simple fractions into which the given fraction is resolvable.

6. It is required to convert the rational fraction $\frac{1+x^2}{x-x^3}$ into its equivalent simple fractions.

Here the factors of the denominator $x-x^3$ being x, $1-x$, and $1+x$, we shall have

$$\frac{1+x^2}{x-x^3}=\frac{A}{x}+\frac{A'}{1-x}+\frac{A''}{1+x},$$

as in example 2; whence, for x, the first factor, $A=\frac{N}{S}=\frac{1+x^2}{1-x^2}=1$, x being $=0$; and for $1-x$, the second factor, $A'=\frac{N}{S}=\frac{1+x^2}{x+x^2}=1$, x being $=1$; also, for $1+x$, the third factor, $A''=\frac{N}{S}=\frac{1+x^2}{x-x^2}=-1$, x being $=-1$; therefore $\frac{1+x^2}{x-x^3}=\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}$, the simple fractions, as before.

7. It is required to convert the rational fraction $\frac{1}{x^3(1-x)^2(1+x)}$ into its equivalent simple fractions.

Here, the factors of the denominator being x^3, $(1-x)^2$, and $1+x$, we shall have

$$\frac{1}{x^3(1-x)^2(1+x)}=\frac{A}{1+x}+\frac{B}{x^3}+\frac{B'}{x^2}+\frac{B''}{x}+\frac{C}{(1-x)^2}+\frac{C'}{1-x};$$

whence, for $1+x$, the first factor, $A=\frac{N}{S}=\frac{1}{x^3-2x^4+x^5}=-\frac{1}{4}$, x being $=-1$; and for x^3, the second factor, $B=\frac{N}{S}=\frac{1}{1-x-x^2+x^3}=1$, x being $=0$.

Let now $Q=\frac{N-BS}{x-p}=\frac{x+x^2-x^3}{x}=1+x-x^2$;

then, for x^2, $B'=\frac{Q}{S}=\frac{1+x-x^2}{1-x-x^2+x^3}=1$, x being $=0$.

Let $Q' = \frac{Q - B'S}{x - p} = \frac{2x - x^3}{x} = 2 - x^2$;

then, for x, $B'' = \frac{Q'}{S} = \frac{2 - x^2}{1 - x - x^2 + x^3} = 2$, x being $= 0$.

Also, for $(1 - x)^2$, $C = \frac{N}{S} = \frac{1}{x^3 + x^4} = \frac{1}{2}$, x being $= 1$.

Let now $R = \frac{N - CS}{x - p} = \frac{1 - \frac{1}{2}x^3 - \frac{1}{2}x^4}{x - 1} = 1 + x + x^2 + \frac{1}{2}x^3$;

then, for $1 - x$, $C' = \frac{R}{S} = \frac{1 + x + x^2 + \frac{1}{2}x^3}{x^3 + x^4} = \frac{7}{4}$, x being $= 1$.

Therefore

$$\frac{1}{x^3(1-x)^2(1+x)} = \frac{1}{x^3} + \frac{1}{x^2} + \frac{2}{x} + \frac{1}{2(1-x)^2} + \frac{7}{4(1-x)} - \frac{1}{4(1+x)}.$$

8. Let $\frac{5 - x}{1 - x^2}$ be reduced to its equivalent simple fractions. Ans. $\frac{2}{1 - x} + \frac{3}{1 + x}$

9. Let $\frac{1 - x - 4x^2}{1 - 4x^2 + 3x^3}$ be reduced to its equivalent simple fractions. Ans. $\frac{1}{x} + \frac{2}{1 - x} + \frac{1}{1 + 3x}$

10. Let $\frac{13 + 21x + 2x^2}{1 - 5x^2 + 4x^4}$ be reduced to its equivalent simple fractions. Ans. $\frac{1}{1 + x} - \frac{6}{1 - x} + \frac{2}{1 + 2x} + \frac{16}{1 - 2x}$

11. Let $\frac{1 + 5x + 3x^2}{(1 + x)^2(1 + 2x)^2}$ be reduced to its equivalent simple fractions.

Ans. $\frac{1}{(1 + x)^2} + \frac{3}{1 + x} + \frac{5}{(1 + 2x)^2} - \frac{6}{1 + 2x}$

12. Let $\frac{2 + 28r - x^2}{x(1 + x)(2 - x)(1 - 2x)^2}$ be reduced to its equivalent simple fractions.

Ans. $\frac{1}{x} + \frac{1}{1 + x} + \frac{1}{2 - x} + \frac{2}{1 - 2x} + \frac{14}{(1 - 2x)^2}$

OF RECURRING SERIES.

(K) A RECURRING SERIES is a rank, or progression of quantities, so constituted, that each term has a constant relation to some given number of the preceding terms, taken continually in the same order.

Thus, if $1 + 6x + 12x^2 + 48x^3 + 120x^4$ &c. be the given series, we shall have the 3d term $12x^2 = x \times$ 2d term $+ 6x^2 \times$ 1st term; the 4th term $48x^3 = x \times$ 3d term $+ 6x^2 \times$ 2d term; and so on.

In which case, the compound expression x, $6x^2$, or simply 1, 6, is called the *scale of relation* of the several terms.

And if $1 + 4x + 6x^2 + 11x^3 + 28x^4 + 63x^5$ &c. be the given equation, we shall have the 4th term $11x^3 = 2x \times$ 3d term $- x^2 \times$ 2d term $+ 3x^3 \times$ 1st term; the 5th term $28x^4 = 2x \times$ 4th term $- x^2 \times$ 3d term $+ 3x^3 \times$ 2d term; and so on.

Where $2x$, $-x^2$, $+3x^3$, or 2, −1, +3, is the scale of relation; the numbers composing it being the multipliers by which the several terms of the series, or their coefficients, are produced (s).

(s) It may be here observed, that series of this kind arise from the development of certain fractional expressions, of the form

$$\frac{a}{1-a'x}, \quad \frac{a+bx}{1-a'x-b'x^2}, \quad \frac{a+bx+cx^2}{1-a'x-b'x^2-c'x^3}, \quad . \quad . \quad . \quad .$$

$$\frac{a+bx+cx^2+dx^3+ \quad . \quad . \quad . \quad . \quad px^{m-1}}{1-a'x-b'x^2-c'x^3- \quad . \quad . \quad . \quad . \quad -qx^m};$$

in which the terms of each denominator, abating unity, constitute the scale of relation of the terms of the series, produced from the expansion of the fraction in question.

Thus, if A, B, C, &c. be made to denote the penultimate co-

PROBLEM.

Having given an infinite recurring series, of the form $a + bx + cx^2 + dx^3 + ex^4$ &c. of which the scale of relation of the terms is known, to find its sum.

RULE.

1. When the scale of relation consists only of two terms, as $+a'$, $+b'$, the radix, or sum (s) of the series, infinitely continued, will be

$$s = \frac{a + (b - aa')x}{1 - a'x - b'x^2}.$$

2. When the scale of relation consists of three terms, $+a'$, $+b'$, $+c'$, the sum of the series, infinitely continued, will be

$$s = \frac{a + (b - aa')x + (c - ba' - ab')x^2}{1 - a'x - b'x^2 - c'x^3}.$$

3. In like manner, when the scale of relation

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efficients of the successive terms of the series arising from the development of the first of these fractions, we shall have

$$\frac{a}{1 + a'x} = a - a'\text{A}x - a'\text{B}x^2 - a'\text{C}x^3 - a'\text{D}x^4 \ . \ . \ . \ . \ a'\text{P}x^{n-1}.$$

Where it is evident, that the coefficient of each term is formed by multiplying that immediately preceding it by $-a'$, or the entire term by $-a'x$; in which case, therefore, $-a'$, or $-a'x$, is the scale of relation of the several terms of the series.

Again, if the second of these fractions be converted, in like manner, into a series, we shall have

$$\frac{a + bx}{1 + a'x + b'x} = a + (b + a'\text{A})x - (b'\text{A} + a'\text{B})x^2 - (b'\text{B} + a'\text{C})x^3 - (b'\text{C} + a'\text{D})x^4 \text{ \&c.}$$

Where each coefficient, beginning with the third, is determined by means of the two that precede it, multiplied respectively by consists of four terms, $+a'$, $+b'$, $+c'$, $+d'$, the sum of the series, infinitely continued, will be

$$s=\frac{a+(b-aa')x+(c-ba'-ab')x^2+(d-ca'-bb'-ac')x^3}{1-a'x-b'x^2-c'x^3-d'x^4}.$$

And so on, for any greater number of terms, observing to change the signs of the quantities a', b', c', &c. which are here supposed positive, from $-$ to $+$, when any of them are negative(*t*).

EXAMPLES.

1. Required the sum of the infinite recurring series $1+6x+12x^2+48x^3+120x^4$ &c. the scale of relation of its coefficients being 1, 6.

Here $a=1$, $b=6$, $a'=1$, and $b'=6$; whence, by the rule,

$$s=\frac{a+(b-aa')x}{1-a'x-b'x^2}=\frac{1+(6-1\times 1)x}{1-x-6x^2}=\frac{1+5x}{1-x-6x^2}=\text{sum of the series.}$$

2. Required the sum of the infinite recurring

$-b'$, $-a'$; or each entire term, by the two preceding terms, multiplied by $-b'x$, $-a'x^2$; the quantities $-b'$ $-a'$, or $-b'x$, $-a'x^2$, constituting, in this case, the scale of relation of the terms of the series.

And a similar mode of proceeding may be employed in the development of either of the other fractions; in which cases, however, as well as in those given above, the operation may be more perspicuously performed by the method of indeterminate coefficients, used in Art. c.

(*t*) It is sometimes difficult to determine whether a given series be recurrent or not; in which case, the following rule, first given for that purpose by Lagrange, in the Mem. de l'Acad. des Sciences de Paris, année 1772, will be found useful.

Divide unity, or 1, by the sum of the proposed series, s, as far series $1 + 4x + 6x^2 + 11x^3 + 28x^4 + 63x^5$ &c. the scale of relation of the coefficients being 2, -1, 3.

Here $a=1$, $b=4$, $c=6$, $a'=2$, $b'=-1$, and $c'=3$; whence, by the second case of the rule,

$$s = \frac{a+(b-aa')x+(c-ba'-ab')x^2}{1-a'x-b'x^2-c'x^3} = \frac{1+(4-2)x+(6-8+1)x^2}{1-2x+x^2-3x^3}$$

$$= \frac{1+2x-x^2}{1-2x+x^2-3x^3} = \frac{(1+x)^2-2x^2}{(1-x)^2-3x^3} = \text{sum of the series.}$$

3. Required the sum of the infinite recurring series $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5$ &c. the scale of relation of the coefficients being 2, -1.

Ans. $\frac{1}{(1-x)^2}$

4. Required the sum of the infinite recurring series $1 - \frac{3}{x} + \frac{5}{x^2} - \frac{7}{x^3} + \frac{9}{x^4}$ &c. the scale of relation of the coefficients being -2, -1. Ans. $\frac{x(x-1)}{(x+1)^2}$

5. Required the sum of the infinite recurring series $1 + 3x + 7x^2 + 13x^3 + 25x^4 + 51x^5 + 103x^6$ &c. the scale of relation of the coefficients being $+2$, -1, $+2$. Ans. $\frac{1+x+2x^2}{(1-x)^2-2x^3}$

6. Required the sum of the infinite recurring series $1 + 8x + 27x^2 + 64x^3 + 125x^4 + 216x^5 + 343x^6$ &c. the scale of relation being 4, -6, 4, -1.

Ans. $\frac{1+4x+x^2}{(1-x)^4}$

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as to two terms in the quotient, which let be $p + qx$, and call the remainder $s'x^2$.

Divide, in like manner, s by s', as far as to two terms in the quotient, which let be $p' + q'x$, and call the remainder $s''x^2$.

Do the same with s' and s'', calling the first two terms of the quotient $p'' + q''x$, and the remainder $s'''x^3$, and so on; then, if the series is recurrent, the operation will at length terminate, otherwise not.

Note. When the sum of a limited number of terms only, of a recurring series, is required, it may be determined, by first finding the sum of all the terms of the series, following the term proposed, and then subtracting the result from the whole sum, found by the former rule. Thus (u),

7. Let it be required to find the sum of n terms of the recurring series

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 \; . \; . \; . \; . \; nx^{n-1}.$$

Here, the terms following nx^{n-1}, will be $(n+1)x^n + (n+2)x^{n+1} + (n+3)x^{n+2}$ &c. where the scale of relation of the coefficients is $2, -1$, as before; whence, by substituting $(n+1)x^n$, and $(n+2)x^{n+1}$, for a and bx in the first case of the rule, and making $a' = 2$, and $b' = -1$, we shall have

$$\frac{(n+1)x^n + (n+2)x^{n+1} - 2(n+1)x^{n+1}}{1-2x+x^2} = \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2} = \text{sum}$$

of the series, $(n+1)x^n + (n+2)x^{n+1} + (n+3)x^{n+2} + (n+4)x^{n+3}$ &c. continued *ad infinitum*.

But from the answer to example 3, it appears that the sum of the whole infinite series $1 + 2x + 3x^2 + 4x^3 + 5x^4$ &c. $= \frac{1}{(1-x)^2}$; whence $\frac{1}{(1-x)^2} - \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2}$

(u) This part of the above problem, is too operose and complicated to be insisted upon at much length in a work like the present; the finding the general term of many recurring series, that might be proposed, being frequently attended with considerable difficulties; particularly in determining the several simple fractions, that are deducible from the radix, or generative fraction, which is a necessary part of the operation.

For this purpose, therefore, as well as for more ample details on the subject, see the *Introductio in Analysin infinitorum* of Euler, or the French translation of that work, with notes, by Labey: also the Miscell. Analytica of Demoivre.

$= \dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2} =$ sum of n terms of the same series $1+2x+3x^2+4x^3+5x^4 \; . \; . \; . \; . \; nx^{n-1}$.

8. Required the sum of n terms of the infinite recurring series $1+4x+14x^2+46x^3+146x^4+454x^5 \; . \; . \; . \; . \; (2.3^n-2^n)x^n$, the scale of relation of the coefficients being $5, -6$.

Ans. $\dfrac{1-x-(2.3^n-2^n)x^n+(4.3^n-3.2^n)x^{n+1}}{1-5x+6x^2}$

9. Required the sum of n terms of the infinite recurring series $1+0x+2x^2+2x^3+6x^4+10x^5+22x^6+42x^7+86x^8 \; . \; . \; . \; . \; \dfrac{2^n \pm 2}{3}x^n$, using the upper sign when n is an even number, and the under sign when it is odd; and observing that the scale of relation is 1, 2. Ans. $\dfrac{1-x-\frac{1}{3}(2^n \pm 2)x^n-\frac{1}{3}(2^n \mp 4)x^{n+1}}{1-x-2x^2}$

10. Required the sum of n terms of the series $1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7 \; . \; . \; . \; . \; \dfrac{2n+3 \pm 1}{4}x^n$, using the upper sign when n is an even number, and the under sign when it is odd; and observing that the scale of relation is $1, 1, -1$.

OF THE SUMMATION AND INTERPOLATION OF INFINITE SERIES.

(L) **AN INFINITE SERIES** is a continued rank, or progression of quantities, connected together by the signs + or −; being such as usually proceeds according to some regular law (x).

(x) The doctrine of infinite series is a subject which has engaged the attention of the greatest mathematicians, both ancient

$$\text{As } \frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\&c.$$

$$\text{Or } \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\&c.$$

In the first of which, the several terms are the reciprocals of the odd numbers 1, 3, 5, 7, &c.; and in the latter, the reciprocals of the even numbers 2, 4, 6, 8, &c. with alternate signs.

Series of this kind are also of different forms, and arise in various ways; as, from the common operations of division, the extraction of roots, and in many other processes of calculation.

Thus, if 1 be divided by $a+b$, or $a-b$, the quotient, in each of these cases, will be as below:

$$\frac{1}{a+b}=\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}-\&c.$$

$$\frac{1}{a-b}=\frac{1}{a}+\frac{b}{a^2}+\frac{b^2}{a^3}+\frac{b^3}{a^4}+\frac{b^4}{a^5}+\&c.$$

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and modern; and, when taken in its whole extent, is, perhaps, one of the most abstruse and difficult branches of abstract mathematics.

To find the sum of a series, the number of whose terms is inexhaustible, or infinite, has been regarded, by some, as a paradox, or a thing impossible to be done; but this difficulty will be easily removed, by considering that every finite magnitude whatever is divisible *in infinitum*, or consists of an indefinite number of parts, the aggregate, or sum, of which, is equal to the quantity first proposed.

A number actually infinite is, indeed, a plain contradiction to all our ideas; for any number that we can possibly conceive, or of which we have any notion, must always be determinate and finite; so that a greater may still be assigned, and a greater after this; and so on, without a possibility of ever coming to an end of the increase or addition.

This inexhaustibility, therefore, in the nature of numbers, is all that we can distinctly comprehend by their infinity;-for though

Where the terms of each series are the same, except that the signs are alternately positive and negative in the former, and all positive in the latter.

And, if *a* or *b* be expounded by 1, 2, 3, &c. we shall obtain the following numeral series, and their values.

$$\frac{1}{1+1}=\frac{1}{2}=1-1+1-1+1-\&c.$$

$$\frac{1}{3-1}=\frac{1}{2}=\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\frac{1}{3^5}+\&c.$$

$$\frac{1}{2+1}=\frac{1}{3}=\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^5}-\&c.$$

$$\frac{1}{1+2}=\frac{1}{3}=1-2+2^2-2^3+2^4-\&c.$$

$$\frac{1}{1-\frac{1}{2}}=2=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\&c.$$

&c. &c.

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we can easily conceive, that a finite quantity may become greater and greater without end, yet we are not, by that means, enabled to form any notion of the *ultimatum*, or last magnitude, which is incapable of farther augmentation.

Hence, we cannot apply to an infinite series the common notion of a sum, or a collection of several particular numbers, which are joined and added together, one after another: for this supposes, that each of the numbers, composing that sum, is known and determined. But as every series generally observes some regular law, and continually approaches towards a term, or limit, we can easily conceive it to be a whole, of its own kind, and that it must have a certain real value, whether that value be determinable or not.

Thus, in many series, a number is assignable, beyond which no number of its terms can ever reach, or, indeed, be ever perfectly equal to it; but yet may approach towards it in such a manner, as to differ from it by less than any quantity that can be named. So that we may justly call this the value or sum of the series; not as being a number found by the common method of addition, Whence it appears that the same quantity may be expressed by different series; and, on the contrary, that different series may have the same value.

In like manner it may also be shown, by the common methods of extracting the square, cube, &c. root, or by the binomial theorem, that

$$(1+1)^{\frac{1}{2}} = \sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2.4} + \frac{1.3}{2.4.6} - \frac{1.3.5}{2.4.6.8} + \&c.$$

$$(1-1)^{\frac{1}{2}} = 0 = 1 - \frac{1}{2} - \frac{1}{2.4} - \frac{1.3}{2.4.6} - \frac{1.3.5}{2.4.6.8} - \&c.$$

$$(1+1)^{\frac{1}{3}} = \sqrt[3]{2} = 1 + \frac{1}{3} - \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} + \&c.$$

$$(1-1)^{\frac{1}{3}} = 0 = 1 - \frac{1}{3} - \frac{2}{3.6} - \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} - \&c.$$

&c. &c.

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but such a limitation of the value of the series, taken in all its infinite capacity, that, if it were possible to add all the terms together, one after another, the sum would be equal to that number.

In other series, on the contrary, the aggregate, or value of the several terms, taken collectively, has no limitation; which state of it may be expressed by saying, that the sum of the series is infinitely great; or, that it has no determinate or assignable value, but may be carried on to such a length, that its sum shall exceed any given number whatever.

Thus, as an illustration of the first of these cases, it may be observed, that, if r be the ratio, g the greatest term, and l the least, of any decreasing geometric series, the sum, according to the common rule, will be $(rg-l) \div (r-1)$; and if we suppose the less extreme, l, to be diminished till it becomes $=0$, the sum of the whole series will be $rg \div (r-1)$: for it is demonstrable, that the sum of no assignable number of terms of the series can ever be equal to that quotient; and yet no number less than it will ever be equal to the value of the series.

Whatever consequences, therefore, follow from the supposition of $rg \div (r-1)$ being the true and adequate value of the series, taken

From which it is evident, that the aggregate value of all the terms of various kinds of infinite series may be $=0$; or that the first term of such a series may be greater than, or equal to, the sum of all the following terms.

It may also be remarked, as a curious circumstance, with respect to the infinite series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\text{\&c.}$$

of which the value is 2, that any one of its terms is equal to the sum of all those that succeed it.

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in all its infinite capacity, as if all the parts were actually determined, and added together, no assignable error can possibly arise from them, in any operation or demonstration where the sum is used in that sense; because, if it should be said that the series exceeds that value, it can be proved, that this excess must be less than any assignable difference; which is, in effect, no difference at all; whence the supposed error cannot exist, and consequently $rg \div (r-1)$ may be looked upon as expressing the true value of the series, continued to infinity.

We are, also, farther satisfied of the reasonableness of this doctrine, by finding, in fact, that a finite quantity is frequently convertible into an infinite series, as appears in the case of circulating decimals. Thus $\frac{2}{3}$, expressed decimally, is $=66666$ &c.

$$=\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}\text{\&c.}$$

continued *ad infinitum*. But this is a geometric series, the first term of which is $\frac{6}{10}$, and the ratio $\frac{1}{10}$; and therefore the sum of all its terms, continued to infinity, will evidently be equal to $\frac{2}{3}$, or the number from which it was originally derived. And the same may be shown of many other series, and of all circulating decimals in general.

To this we may likewise add, that series receive particular names, according to the nature of their formation, or other circumstances attending them. Thus:

A converging series, is that in which the terms decrease, or become continually less and less; being so called, because the successive sums of its several terms approximate, or converge, nearer and nearer to the true value of the series (*y*).

Thus, in the infinite geometric series before given,

$$\frac{1}{1-\frac{1}{2}}=2=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}\&c.$$

where all the signs are +, the first term $=1$; the sum of the first two terms, $1+\frac{1}{2}=1\frac{1}{2}$; the sum of the first three, $1+\frac{1}{2}+\frac{1}{4}=1\frac{3}{4}$; the sum of the first four, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=1\frac{7}{8}$; and so on: where each result approaches nearer to 2, the true value of the series, than that immediately preceding it.

In like manner, if the terms of the series be al-

(*y*) It may be here proper to observe, that the sum of a series of continually decreasing fractions is not always finite; as is well known to be the case with the reciprocals of the natural numbers $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\&c.$ the value of which, exceeds any assignable number whatever; though, at the same time, the sum of the reciprocals of the squares of the same numbers, $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}\&c.$ is finite; being equal to $\frac{1}{6}$ of the square of the circumference of the circle, whose diameter is 1; as is shown by Euler, in the Acta Petrop, vol. vii. and by Maclaurin, in his Fluxions, Art. 822.

ternately positive and negative, the successive sums, in this case, will also be alternately too great and too little; though still approaching nearer and nearer to the final sum, or true value of the series.

Thus, taking the same series as before, but with alternate signs,

$$\frac{1}{1+\frac{1}{2}}=\frac{2}{3}=1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}-\&c.$$

we shall have, the first term $=1$; the first two terms, $1-\frac{1}{2}=\frac{1}{2}$; the first three, $1-\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$; the first four, $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}=\frac{5}{8}$; and so on: where the first result, 1, is too great; the second, $\frac{1}{2}$, too little; the third, $\frac{3}{4}$, too great; and the fourth, $\frac{5}{8}$, too little; but each nearer to $\frac{2}{3}$ than the former.

A diverging series, is that in which the terms increase, or become continually greater and greater; being so called, because the successive sums of its terms diverge, or go farther and farther from the true value of the series; so as to be all too great when the terms are all positive, and alternately too great and too little, when they are positive and negative by turns.

Thus, in the infinite diverging series, before given,

$$\frac{1}{1+2}=\frac{1}{3}=1-2+4-8+16-\&c.$$

the first term $=1$; the first two terms, $1-2=-1$; the first three, $1-2+4=3$; the first four, $1-2+4-8=-5$; and so on: where the results are alternately too great and too little, and diverging more and more from $\frac{1}{3}$, the true value of the series.

A neutral series, is that in which all the terms are equal to each other, but with alternate signs; being so called, because the successive sums of the terms, in this case, are always at the same distance from the true value of the series, but alternately positive and negative, or too great and too little.

Thus, in the following neutral series,

$$\frac{1}{1+1} = \frac{1}{2} = 1 - 1 + 1 - 1 + 1 - \&c.$$

the successive sums of the several terms are, evidently, 1, 0, 1, 0, &c.; which are equally distant from $\frac{1}{2}$, the true value of the series; the one being as much above it, as the other is below it (z).

An ascending series, is that in which the powers of the indeterminate, or unknown quantity, continually increase; as

$$1 + ax + bx^2 + cx^3 + dx^4 + \&c.$$

A descending series, is that in which the powers of the indeterminate quantity continually decrease

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(z) A number of controversies have arisen, at different times, about the nature of neutral and diverging series, which have served more to perplex the subject, than to throw any new light upon the matter in dispute; which, indeed, requires no particular elucidation; as these expressions are merely the developments of certain formulæ, to which they must necessarily be equivalent, whatever embarrassments may attend the consideration of them when taken separately.

Leibnitz was one of the first who treated of the neutral series $1 - 1 + 1 - 1 + 1 - 1 +$ &c.; and concluded, by an abstract method of reasoning, derived from the series itself, that the sum of an infinite number of its terms must be $= \frac{1}{2}$; but his opponents justly objected to the arguments he had used, as being inconcluin the numerators, or increase in the denominators;

$$\text{As, } 1 + ax^{-1} + bx^{-2} + cx^{-3} + dx^{-4} + \&c.$$

$$\text{Or, } 1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{d}{x^4} + \&c.$$

The supplement of a series, obtained by division, is the quantity that arises from dividing what remains, after any particular term, by the divisor; and if this be combined with the foregoing terms, the series will be rendered finite, and equal in value to the whole. Thus,

$$\frac{1}{2+1} = \frac{1}{3} = \frac{1}{2} - \frac{1}{6} = \frac{1}{2} - \frac{1}{4} + \frac{1}{12} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{24}\&c.$$

The law of continuation of a series, is a certain relation that subsists between the coefficients of its several terms; which is sometimes obvious, and in many other cases can be found by dividing each of the coefficients by that immediately preceding it. Thus, in the series

$$1 + \frac{2}{3}x + \frac{8}{15}x^2 + \frac{16}{35}x^3 + \frac{128}{315}x^4\&c.$$

by dividing the second term by the first, the third by the second, the fourth by the third, and so on, we shall obtain the quotients

$$\frac{2}{3}, \quad \frac{4}{5}, \quad \frac{6}{7}, \quad \frac{8}{9}\&c.$$

Whence, if these numbers be successively multiplied together, according to the order in which

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sive; and it may be fairly questioned whether they would have led him to the right result, if he had not previously known what it was.

Among other speculations on the subject, taking it in this way, it might have been remarked, as a curious circumstance, that the sum of a series of whole numbers should be equal to a fraction; but these difficulties vanish, when, at any particular step of the operation, the remainder, arising from dividing the numerator by the denominator, is taken into consideration.

they stand, the above series may be expressed as follows:

$$1+\frac{2}{3}x+\frac{2.4}{3.5}x^2+\frac{2.4.6}{3.5.7}x^3+\frac{2.4.6.8}{3.5.7.9}x^4\&c.$$

Where the law, by which it may be indefinitely continued, is sufficiently manifest.

A summable series, is that of which the aggregate value, or sum of all its terms, can be accurately determined. As, for instance, the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\&c.$$

which has for its limit, or the sum of its terms, unity or 1.

And the same may be shown of a great number of different series, the sums of which can be exactly ascertained; while, on the contrary, the sums of many others are infinite, or such as can only be determined by approximation (*a*).

(*a*) Among other circumstances relating to this subject, it may be observed, that a series often converges so slowly as to be of no use in practice. Thus, if it were required to determine the sum of the series

$$\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+\frac{1}{9.10}\&c.$$

which lord Brouncker found for the quadrature of the hyperbola, true to nine places of figures, by the mere addition of its terms, Mr. Sterling has computed, that it would be necessary to add a thousand million of them together, for that purpose; a labour for which the life of man would not be sufficient; but by a method which he has followed, as well as by other means, the approximate sum of the series, to the degree of exactness here mentioned, may be found by a very moderate computation.—See Sterling, Method. Differentialis, page 26.

Newton has also remarked, with respect to the series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}\&c.$$

which is $\frac{1}{8}$ of the circumference of the circle, whose radius is 1, With respect to the processes by which the summation of series of this kind are generally obtained, one of the principal is by the method of differences, pointed out and illustrated in prob. IV. next following.

Another method is that first employed by James and John Bernoulli, which consists in resolving the given series into several others of which the summation is known; or by subtracting from the given series, when put = s, the same series, deprived of some of its first terms, and then, from the value of the remainder, to determine the sum of the series first proposed.

A third method, which is that of Demoivre, consists in putting the sum of the series = s, and multiplying each side of the equation by some binomial or trinomial expression, which involves the powers of the unknown quantity x, and certain known coefficients; then, by taking x so that the assumed binomial, &c. shall become = 0, and transposing some of the first terms, the result will give the sum of the series sought.

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that to exhibit its value true to twenty places of decimals, would require the computation of no less than five thousand millions of its terms, the performance of which would occupy more than a thousand years. — See Newton's second letter to Oldenburgh, in the *Commercium Epistolicum*, page 159.

And the labour attending this operation would be but little abridged, if the series were represented under the form

$$2\left(\frac{1}{3}+\frac{1}{5.7}+\frac{1}{9.11}+\frac{1}{13.15}\&c.\right)$$

to which it can be easily reduced; though by means of series of a similar kind, which have since been discovered by Machin, Euler, and others, the sum may be readily found to a still greater degree of accuracy.

Each of which methods, modified so as to render it more commodious in practice, together with several other artifices for the same purpose, will be found sufficiently elucidated in the miscellaneous questions succeeding the following problems.

PROBLEM I.

Any series being given to find its several orders of differences.

RULE.

1. Take the first term from the second, the second from the third, the third from the fourth, &c. and the remainders will form a new series, called the *first order of differences.*

2. Take the first term of this last series from the second, the second from the third, the third from the fourth, &c. and the remainders will form another new series, called the *second order of differences.*

3. Proceed, in like manner, for the third, fourth, fifth, &c. orders of differences; and so on till they terminate, or are carried as far as may be thought necessary (*b*).

EXAMPLES.

1. Required the several orders of differences of the series 1, 2^2, 3^2, 4^2, 5^2, 6^2, &c.

1, 4, 9, 16, 25, 36, &c.
3, 5, 7, 9, 11, &c. 1st diff.
2, 2, 2, 2, &c. 2d diff.
0, 0, 0, &c.

(*b*) When the several terms of the series continually increase, the differences will be all positive; but when they decrease, the differences will be negative and positive alternately.

2. Required the several orders of differences of the series $1, 2^3, 3^3, 4^3, 5^3, 6^3$, &c.

1, 8, 27, 64, 125, 216, &c.
7, 19, 37, 61, 91, &c. 1st diff.
12, 18, 24, 30, &c. 2d diff.
6, 6, 6, &c. 3d diff.
0, 0, &c.

3. Required the several orders of differences of the series 1, 3, 6, 10, 15, 21, &c.

Ans. 1st, 2, 3, 4, 5, &c.; 2d, 1, 1, 1, &c.

4. Required the several orders of differences of the series 1, 6, 20, 50, 105, 196, &c.

Ans. 1st, 5, 12, 30, 45, 91, &c.; 2d, 9, 16, 25, 36, &c.; 3d, 7, 9, 11, &c.; 4th, 2, 2, &c.

5. Required the several orders of differences of the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, &c.

PROBLEM II.

Any series, *a*, *b*, *c*, *d*, *e*, &c. being given, to find the first term of the *n*th order of differences.

RULE.

Let δ stand for the first term of the *n*th differences.

Then will $a - nb + n \cdot \frac{n-1}{2}c - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}d + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}e$ &c. to $n+1$ terms $= \delta$, when n is an even number.

And $-a + nb - n \cdot \frac{n-1}{2}c + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}d - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{3}e$ &c. to $n+1$ terms $= \delta$, when n is an odd number (c).

(c) When the terms of several orders of differences happen to be very great, it will be more convenient to take the logarithms

EXAMPLES.

1. Required the first term of the third order of differences of the series 1, 5, 15, 35, 70, &c.

Here a, b, c, d, e, &c. $=1, 5, 15, 35, 70$, &c. and $n=3$.

Whence $-a+nb-n.\frac{n-1}{2}c+n.\frac{n-1}{2}.\frac{n-2}{3}d=-a+3b-3c+d=-1+15-45+35=4=$ the first term required.

2. Required the first term of the fourth order of differences of the series, 1, 8, 27, 64, 125, &c.

Here a, b, c, d, e, &c. $=1, 8, 27, 64, 125$, &c. and $n=4$.

Whence $a-nb+n.\frac{n-1}{2}c-n.\frac{n-1}{2}.\frac{n-2}{3}d+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}e=a-4b+6c-4d+e=1-32+162-256+125=0$; so that the first term of the fourth order is 0.

3. Required the first term of the eighth order of differences of the series, 1, 3, 9, 27, 81, &c (d).

Ans. 256

4. Required the first term of the fifth order of differences of the series, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c.

Ans. $-\frac{1}{36}$

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of the quantities concerned, whose differences will be smaller; and, when the operation is finished, the quantity answering to the last logarithm may be easily found.

(d) The labour, in questions of this kind, may be often abridged, by putting ciphers for some of the terms at the beginning of the series; by which means several of the differences will be equal to 0, and the answer, on that account, obtained in fewer terms.

PROBLEM III.

To find the nth term of the series, a, b, c, d, e, &c. when the differences of any order become at last equal to each other.

RULE.

Let d', d'', d''', d^{iv}, &c. be the first of the several orders of differences, found as in the last problem.

Then will $a+\frac{n-1}{1}d'+\frac{n-1}{1}\cdot\frac{n-2}{2}d''+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}d'''+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}\cdot\frac{n-4}{4}d^{iv}$ &c. $=n$th term required.

EXAMPLES.

1. It is required to find the twelfth term of the series, 2, 6, 12, 20, 30, &c.

$$\begin{array}{cccccc}
2, & 6, & 12, & 20, & 30, & \&c. \\
 & 4, & 6, & 8, & 10, & \&c. \\
 & & 2, & 2, & 2, & \&c. \\
 & & & 0, & 0, & \&c.
\end{array}$$

Here 4 and 2 are the first terms of the differences.

Let, therefore, $4=d'$, $2=d''$, and $n=12$.

Then $a+\frac{n-1}{1}d'+\frac{n-1}{1}\cdot\frac{n-2}{2}d''=2+11d'+55d''$ $=2+44+110=156=12$th term, or the answer required.

2. Required the twentieth term of the series, 1, 3, 6, 10, 15, 21, &c.

$$\begin{array}{ccccccc}
1, & 3, & 6, & 10, & 15, & 21, & \&c. \\
 & 2, & 3, & 4, & 5, & 6, & \&c. \\
 & & 1, & 1, & 1, & 1, & \&c. \\
 & & & 0, & 0, & 0, & \&c.
\end{array}$$

Here 2 and 1 are the first terms of the differences.

Let, therefore, $2=d'$, $1=d''$, and $n=20$.

Then $a+\frac{n-1}{1}d'+\frac{n-1}{1}\cdot\frac{n-2}{2}d''=1+19d'+171d''$ $=1+38+171=210=$ 20th term required.

3. Required the fifteenth term of the series, 1, 4, 9, 16, 25, 36, &c. Ans. 225

4. Required the twentieth term of the series, 1, 8, 27, 64, 125, &c.

5. Required the thirtieth term of the series, $1, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}$, &c.

PROBLEM IV (*e*).

To find the sum of *n* terms of the series, *a*, *b*, *c*, *d*, *e*, &c. when the differences of any order become at last equal to each other.

RULE.

Let d', d'', d''', d^{iv}, &c. be the first of the several orders of differences.

Then will $na+n\cdot\frac{n-1}{2}d'+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}d''+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}d'''+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot\frac{n-4}{5}d^{\text{iv}}$ &c. $=$ to the sum of *n* terms of the series.

(*e*) When the differences in this or the former rule are finally $=0$, any term, or the sum of any number of the terms, may be accurately determined; but if the differences do not vanish, the result is only an approximation; which, however, may be often very usefully applied, and will become nearer and nearer the truth as the differences diminish.

EXAMPLES.

1. Required the sum of n terms of the series, 1, 2, 3, 4, 5, 6, &c.

$$\begin{array}{cccccccc}
1, & 2, & 3, & 4, & 5, & 6, & \&c. \\
 & 1, & 1, & 1, & 1, & 1, & \&c. \\
 & & 0, & 0, & 0, & 0, & \&c.
\end{array}$$

Here 1 and 0 are the first terms of the differences.

Let, therefore, $a=1$, $d'=1$, and $d''=0$.

Then will $na+n.\frac{n-1}{2}d'=n+\frac{n^2-n}{2}=\frac{n^2+n}{2}=$ sum of n terms, as required.

2. Required the sum of n terms of the series, 1^2, 2^2, 3^2, 4^2, 5^2, &c.; or 1, 4, 9, 16, 25, &c.

$$\begin{array}{cccccc}
1, & 4, & 9, & 16, & 25, & \&c. \\
 & 3, & 5, & 7, & 9, & \&c. \\
 & & 2, & 2, & 2, & \&c. \\
 & & & 0, & 0, & \&c.
\end{array}$$

Here 3 and 2 are the first terms of the differences.

Let, therefore, $a=1$, $d'=3$, and $d''=2$.

Then will $na+n.\frac{n-1}{2}d'+n.\frac{n-1}{2}.\frac{n-2}{3}d''=n+3n.\frac{n-1}{2}+2n.\frac{n-1}{2}.\frac{n-2}{3}=\frac{3n^2-3n}{2}+\frac{n^3-3n^2+2n}{3}=\frac{n\times(n+1)\times(2n+1)}{6}=$ sum of n terms, as required.

3. Required the sum of n terms of the series, 1^3, 2^3, 3^3, 4^3, 5^3, &c.; or 1, 8, 27, 64, 125, &c.

$$\begin{array}{cccccc}
1, & 8, & 27, & 64, & 125, & \&c. \\
 & 7, & 19, & 37, & 61, & \&c. \\
 & & 12, & 18. & 24, & \&c. \\
 & & & 6, & 6, & \&c. \\
 & & & & 0, & \&c.
\end{array}$$

Hence the first terms of the differences are 7, 12, and 6.

Let, therefore, $a=1$, $d'=7$, $d''=12$, and $d'''=6$.

Then will $na+n\,.\,\frac{n-1}{2}d'+n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}d''+n\,.$ $\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,\frac{n-3}{4}d'''=n+7n\,.\,\frac{n-1}{2}+12n.\frac{n-1}{2}\,.\,\frac{n-2}{3}+$ $6n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,\frac{n-3}{4}=n+\frac{7n^2-7n}{2}+2n^3-6n^2+$ $4n+\frac{n^4-6n^3+11n^2-6n}{4}=\frac{4n}{4}+\frac{14n^2-14}{4}n+\frac{8n^3-24n^2+16n}{4}+$ $\frac{n^4-6n^3+11n^2-6n}{4}=\frac{n^4+2n^3+n^2}{4}=$ sum of n terms, as required.

4. Required the sum of n terms of the series, 2, 6, 12, 20, 30, &c. Ans. $\frac{n\times(n+1)\times(n+2)}{3}$

5. Required the sum of n terms of the series, 1, 3, 6, 10, 15, &c. Ans. $\frac{n}{1}\,.\,\frac{n+1}{2}\,.\,\frac{n+2}{3}$

6. Required the sum of n terms of the series, 1, 4, 10, 20, 35, &c. Ans. $\frac{n}{1}\,.\,\frac{n+1}{2}\,.\,\frac{n+2}{3}\,.\,\frac{n+3}{4}$

7. Required the sum of n terms of the series, 1^4, 2^4, 3^4, 4^4, &c.; or 1, 16, 81, 256, &c.

Ans. $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$

PROBLEM V.

The series a, b, c, d, e, &c. being given, whose terms are an unit's distance from each other, to find any intermediate term by interpolation.

RULE.

Let x be the distance of any term y to be interpolated, and d', d'', d''', &c. the terms of the differences.

Then will $a+xd'+x.\frac{x-1}{2}d''+x.\frac{x-1}{2}.\frac{x-2}{3}d'''+x.\frac{x-1}{2}.\frac{x-2}{3}.\frac{x-3}{4}d^{iv}$ &c. $=y$.

EXAMPLES.

1. Given the logarithmic sines of 1° 0′, 1° 1′, 1° 2′, and 1° 3′, to find the log. sine of 1° 1′ 40″.

Here	1° 0′	1° 1′	1° 2′	1° 3′
Sines	8.2418553	8.2490332	8.2560943	8.2630424
		71779	70611	69481
			−1168	−1130
				38

Whence the first terms of the differences are 71779, −1168, and 38.

Let, therefore, $x=1°\,1'\,40''-1°\,0'=1'40''=1\frac{2}{3}=$ distance of y, the term to be interpolated; and $d'=71779$, $d''=-1168$, and $d'''=38$.

Then will $y=a+xd'+x.\frac{x-1}{2}d''+x.\frac{x-1}{2}.\frac{x-2}{3}d'''$ $=a+\frac{5}{3}d'+\frac{5}{9}d''-\frac{5}{81}d'''=8.2418553+.0119631+.0000694-.0000002=8.2538876=$ sine of 1° 1′ 40″, as was required.

2. Given the series $\frac{1}{50}$, $\frac{1}{51}$, $\frac{1}{52}$, $\frac{1}{53}$, $\frac{1}{54}$, &c. to find the term which stands in the middle between $\frac{1}{52}$ and $\frac{1}{53}$. Ans. $\frac{1}{105}$

3. Given the natural tangents of 88° 54′, 88° 55′, 88° 56′, 88° 57′, 88° 58′, 88° 59′, to find the tangent of 88° 58′ 11″. Ans. 55.711144

PROBLEM VI.

Having given a series of equidistant terms, *a, b, c, d, e,* &c. whose first differences are small; to find any intermediate term by interpolation.

RULE.

Find the value of the unknown quantity in the equation which stands against the given number of terms, in the following table, and it will give the term required (*f*).

1. $a-b=0$
2. $a-2b+c=0$
3. $a-3b+3c-d=0$
4. $a-4b+6c-4d+e=0$
5. $a-5b+10c-10d+5e-f=0$
6. $a-6b+15c-20d+15e-6f+g=0$
7. $a-nb+n\,.\,\frac{n-1}{2}c-n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}d+$ $n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,\frac{n-3}{4}e\,\&c.=0$

EXAMPLES.

1. Given the logarithms of 101, 102, 104, and 105, to find the logarithm of 103.

Here the number of terms are 4.

Therefore, against 4, in the table, we have $a-4b+6c-4d+e=0$; or $c=\frac{4\times(b+d)-(a+e)}{6}=$ value of the unknown quantity, or term to be found.

(*f*) The more terms are given, in any series of this kind, the more accurately will the equation approximate towards the true result.

$$\text{Whence}\left|\begin{array}{l} a=2.0043214\\ b=2.0086002\\ d=2.0170333\\ e=2.0211893 \end{array}\right.$$

$$\begin{array}{r} 4\times(b+d)=16.1025340\\ +e=\ \ 4.0255107\\ \hline 6)12.0770233\\ \hline 2.0128372=\text{log. of } 103, \end{array}$$

as required.

2. Given the cube roots of 45, 46, 47, 48, and 49, to find the cube root of 50. Ans. 3.684031

3. Given the logarithms of 50, 51, 52, 54, 55, and 56, to find the logarithm of 53.

Ans. 1.7242758695

PROMISCUOUS EXAMPLES RELATING TO SERIES.

1. To find the sum (s) of n terms of the series, 1, 2, 3, 4, 5, 6, &c.

First, $1+2+3+4+5$ &c. . . . . . $n=$ s.

And $n+(n-1)+(n-2)+(n-3)+(n-4)$ &c. . . . . $+1=$ s.

Therefore, by addition, $(n+1)+(n+1)+(n+1)+(n+1)$ &c. . . . . . $+(n+1)=2$s.

And consequently $n(n+1)=2$s; or s $=\frac{n^2+n}{2}=$ sum required.

2. To find the sum (s) of n terms of the series, 1, 3, 5, 7, 9, 11, &c.

First, $1+3+5+7+9$ &c. . . . . $(2n-1)=$ s.

And $(2n-1)+(2n-3)+(2n-5)+(2n-7)+(2n-9)$ &c. . . . . . $+1=$ s.

Therefore, by addition,

$$2n+2n+2n+2n+2n+\&c. \;.\;.\;.\;.\; 2n=2s.$$

And consequently $2n \times n = 2s$; or $s = \frac{2n^2}{2} = n^2 =$ sum required.

3. Required the sum (s) of n terms of the series, $a+(a+d)+(a+2d)+(a+3d)+(a+4d)$ &c.

First, $a+(a+d)+(a+2d)+(a+3d)$ &c. . . . . $\{a+(n-1)d\}=s$.

And $a+(nd-d)+a+(nd-2d)+a+(nd-3d)+a+(nd-4d)$ &c. . . . . $a=s$.

Therefore, by addition, $2a+(nd-d)+2a+(nd-d)+2a+(nd-d)$&c. . . . . $2a+(nd-d)=2s$.

And consequently $(2a+nd-d)\times n=2s$; or $s=(2a+nd-d)\times\frac{n}{2}=$ sum required.

Or the same may be done in a different manner, as follows:

$$a+(a+d)+(a+2d)+(a+3d)+(a+4d)\,\&c.$$

$$= \left| \begin{matrix} (+1+1+1+1+1\,\&c.)\times a \\ (+0+1+2+3+4\,\&c.)\times d \end{matrix} \right| = s.$$

But n terms of $1+1+1+1+1$ &c. $=n$.

And n terms of $0+1+2+3+4$ &c. $=\frac{n\times(n-1)}{2}$.

Whence $s=na+\frac{n\times(n-1)d}{2}=\{2a+d(n-1)\}\times\frac{n}{2}$,

which is the same answer as before.

4. To find the sum (s) of n terms of the series 1, x, x^2, x^3, x^4, &c.

First, $1+x+x^2+x^3+x^4$, &c. . . . . $x^{n-1}=s$.

And $x+x^2+x^3+x^4+x^5$, &c. . . . . $x^n=sx$.

Whence, by subtraction, $-1+x^n=sx-s$.

Or $s=\frac{x^n-1}{x-1}=$ sum required.

And, when x is a proper fraction, the sum of the series, continued *ad infinitum* may be found in the same manner.

Thus, putting $1+x+x^2+x^3+x^4+x^5$, &c. $=s$,

We shall have $x+x^2+x^3+x^4+x^5$, &c. $=sx$.

And consequently $-1=sx-s$; or $s-sx=1$.

Whence $s=\frac{1}{1-x}=$ sum of an infinite number of terms, as was to be found.

5. Required the sum (s) of the circulating decimal .999999 &c. continued *ad infinitum*.

$$\text{First, } .999999 \text{ \&c.} = \frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}\text{\&c.} =$$

$$9\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\text{\&c.}\right)=s.$$

$$\text{Or, } \frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\text{\&c.}=\frac{s}{9}.$$

$$\text{Therefore, } 1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\text{\&c.}=\frac{10s}{9}.$$

$$\text{And consequently } 1=\frac{10s}{9}-\frac{s}{9}=\frac{9s}{9}=s;$$

Whence $s=1=$ sum of the series.

6. Required the sum (s) of the series $a^2+(a+d)^2+(a+2d)^2+(a+3d)^2+(a+4d)^2$, &c. continued to n terms.

$$\begin{array}{ll} \text{First, } & a^2=a^2 \\ & (a+d)^2=a^2+2\times 1ad+1d^2 \\ & (a+2d)^2=a^2+2\times 2ad+4d^2 \\ & (a+3d)^2=a^2+2\times 3ad+9d^2 \\ & (a+4d)^2=a^2+2\times 4ad+16d^2 \\ & \quad\text{\&c.} \qquad\qquad \text{\&c.} \end{array}$$

Whence

$$s=\left|\begin{array}{l} \text{Sum of } n \text{ terms of } (1+1+1+1+\text{\&c.})a^2 \\ +\ \ldots\ \text{ditto of } (0+1+2+3+\text{\&c.})2ad \\ +\ \ldots\ \text{ditto of } (0+1+4+9+\text{\&c.})d^2 \end{array}\right.$$

But n terms of $1+1+1+1+\&c. = n$.

And of $0+1+2+3+\&c. = \frac{n(n-1)}{1\,.\,2}$

Also of $0+1+4+9+\&c. = \frac{n(n-1)(2n-1)}{1\,.\,2\,.\,3}$

Therefore $s = na^2 + n(n-1)ad + \frac{n(n-1)(2n-1)}{1\,.\,2\,.\,3}d^2 =$ the whole sum of the series to n terms.

7. Required the sum (s) of the series $a^3 + (a+d)^3 + (a+2d)^3 + (a+3d)^3 + (a+4d)^3\&c.$ continued to n terms.

First, $a^3 = a^3$

$$\begin{aligned}
(a+d)^3 &= a^3 + 3\times 1a^2d + 3\times 1ad^2 + 1d^3\\
(a+2d)^3 &= a^3 + 3\times 2a^2d + 3\times 4ad^2 + 8d^3\\
(a+3d)^3 &= a^3 + 3\times 3a^2d + 3\times 9ad^2 + 27d^3\\
(a+4d)^3 &= a^3 + 3\times 4a^2d + 3\times 16ad^2 + 64d^3\\
(a+5d)^3 &= a^3 + 3\times 5a^2d + 3\times 25ad^2 + 125d^3\\
&\&c. \qquad\qquad\qquad\qquad \&c.
\end{aligned}$$

Whence

$$s = \left|\begin{array}{l}
\text{Sum of } n \text{ terms of } (1+1+1+1\&c.)a^3\\
\quad + \ \ldots \ \text{ditto of } (0+1+2+3\&c.)3a^2d\\
\quad + \ \ldots \ \text{ditto of } (0+1+4+9\&c.)3ad^2\\
\quad + \ \ldots \ \text{ditto of } (0+1+8+27\&c.)d^3
\end{array}\right.$$

But n terms of $1+1+1+1\&c. = n$.

Ditto . . . of $0+1+2+3\&c. = \frac{n(n-1)}{1\,.\,2}$

Ditto . . . of $0+1+4+9\&c. = \frac{n(n-1)(2n-1)}{1\,.\,2\,.\,3}$

Ditto . . of $0+1+8+27\&c. = \frac{n^4-2n^3+n^2}{2\times 2}$

Therefore $s = na^3 + \frac{n(n-1)3a^2d}{2} + \frac{n(n-1)(2n-1)3ad^2}{6} + \frac{(n^4-2n^3+n^2)d^3}{4} =$ sum of n terms, as was to be found.

8. Required the sum (s) of n terms of the series $1+3+7+15+31$ &c.

The terms of this series are evidently equal to 1, (1 + 2), (1 + 2 + 4), (1 + 2 + 4 + 8), &c. or the successive sums of the geometrical progression 1, 2, 4, 8, 16, &c.

Let, therefore, $a = 1$ and $r = 2$, and we shall have $a + ar + ar^2 + ar^3 + ar^4$&c. $= 1 + 2 + 4 + 8 + 16$ &c.

But the sum of 1, 2, 3, 4, &c. terms of this series are,

$$1.\ \frac{ar - a}{r - 1} = (r - 1) \times \frac{a}{r - 1}$$

$$2.\ \frac{ar^2 - a}{r - 1} = (r^2 - 1) \times \frac{a}{r - 1}$$

$$3.\ \frac{ar^3 - a}{r - 1} = (r^3 - 1) \times \frac{a}{r - 1}$$

$$4.\ \frac{ar^4 - a}{r - 1} = (r^4 - 1) \times \frac{a}{r - 1}$$

&c. &c.

$$\text{Therefore } s = \frac{a}{r - 1} \times \left| \begin{array}{l} n \text{ terms of } r + r^2 + r^3 + r^4\text{\&c.} \\ -n \text{ terms of } 1 + 1 + 1 + 1\text{\&c.} \end{array} \right.$$

$$\text{But } 1 + 1 + 1 + 1 + 1 + 1 + 1\text{\&c.} = n$$

$$\text{And } r + r^2 + r^3 + r^4 + \text{\&c.} = (r^n - 1) \times \frac{r}{r - 1}$$

Whence $s = \frac{r(r^n - 1)}{r - 1} \times \frac{a}{r - 1} - n \times \frac{a}{r - 1} = 2(2^n - 1) - n =$ whole sum required.

9. Required the sum of n terms of the series $\frac{1}{1} + \frac{3}{2} + \frac{7}{4} + \frac{15}{3} + \frac{31}{16}$&c.

The terms of this series are the successive sums of the geometrical progression $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$&c.

Let, therefore, $a = 1$ and $r = 2$, then will $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$&c. $= a + \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} + \frac{a}{r^4} + \frac{a}{r^5} + \frac{a}{r^6}$&c.

But the sums of 1, 2, 3, 4, &c. terms of this series are,

$$1.\ \frac{(r-1)a}{(r-1)\times 1}=(r-1)\times\frac{a}{r-1}$$

$$2.\ \frac{(r^2-1)\times a}{(r-1)\times r}=\left(r-\frac{1}{r}\right)\times\frac{a}{r-1}$$

$$3.\ \frac{(r^3-1)\times a}{(r-1)\times r^2}=\left(r-\frac{1}{r^2}\right)\times\frac{a}{r-1}$$

$$4.\ \frac{(r^4-1)\times a}{(r-1)\times r^3}=\left(r-\frac{1}{r^3}\right)\times\frac{a}{r-1}$$

&c. &c.

Therefore

$$s=\frac{a}{r-1}\times\left|\begin{array}{l} n \text{ terms of } r+r+r+r \text{ \&c.} \\ -n \text{ terms of } \frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3} \text{ \&c.}\end{array}\right.$$

But $r+r+r+r+r+r$ &c. $=nr$.

And $\frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$ &c. $=\frac{r^n-1}{(r-1)r^{n-1}}$.

Whence

$$s=\frac{a}{r-1}\times(nr-\frac{r^n-1}{(r-1)r^{n-1}}=\frac{(n-1)2^n+1}{2^{n-2}}=\text{sum required.}$$

10. Required the sum (s) of the infinite series of the reciprocals of the triangular numbers,

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}\text{ \&c.}$$

Let $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$ &c. *ad infinitum* $=s$.

Or $\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{2.3}+\frac{1}{2.5}$ &c. . . . . $=s$.

Then $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$ &c. . . . . $=\frac{s}{2}$.

That is, $\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)$ &c. $=\frac{s}{2}$.

$$\text{Or, }\left|\begin{array}{l}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\text{ \&c.}\\ \quad-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}\text{ \&c.}\end{array}\right|=\frac{s}{2}.$$

Whence $\frac{s}{2}=\frac{1}{1}$; or $s=2=$ sum required.

11. And if it be required to find the sum of n terms of the same series, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$ &c.

Let $z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ &c. to $\frac{1}{n}$.

Then $z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ &c. to $\frac{1}{n}$.

And $z-\frac{1}{1}+\frac{1}{n+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ &c. to $\frac{1}{n+1}$.

Therefore $\frac{1}{1}-\frac{1}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$ &c. to $\frac{1}{n}-\frac{1}{n+1}$.

Or $\frac{n}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$ &c. to $\frac{1}{n(n+1)}$.

Whence $\frac{2n}{n+1}=\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$ &c. to $\frac{2}{n(n+1)}$.

Or $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$ &c. to $\frac{2}{n(n+1)}=\frac{2n}{n+1}=$ sum of n terms of the series, as was required.

12. Required the sum of the infinite series, $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$ &c.

Let $z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ &c. *ad infinitum.*

Then $z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ &c. by transposition.

And $1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$ &c. by subtraction.

Or $1-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}$ &c. by transposition.

Whence $\frac{1}{2}=\frac{4}{1.4.3}+\frac{6}{2.9.4}+\frac{8}{3.16.5}$ &c. by subtraction.

Or $\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}$ &c.

And $\frac{1}{2}\div 2=\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$ *ad infinitum.*

But $\frac{1}{2} \div 2 = \frac{1}{4}$; therefore $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}$ continued to infinity, is equal to $\frac{1}{4}$, which is the sum required.

13. And if it were required to find the sum of n terms of the same series $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6}$ &c.

$$\text{Let } z = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \text{\&c. to } \frac{1}{n(n+1)}$$

$$\text{Then } z - \frac{1}{2} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} \text{ \&c. to } \frac{1}{n(n+1)}.$$

And $z - \frac{1}{2} + \frac{1}{(n+1)(n+2)} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8}$ &c. continued to $\frac{1}{(n+1)(n+2)}$ terms.

Therefore $\frac{1}{2} - \frac{1}{(n+1)(n+2)} = \frac{2}{1.2.3} + \frac{2}{2.3.4} + \frac{2}{3.4.5}$ &c. to n terms, by subtraction.

Whence $\frac{1}{4} - \frac{1}{2.(n+1)(n+2)} = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}$ &c. to n terms, by division.

And consequently $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}$ &c. continued to n terms $= \frac{1}{4} - \frac{1}{2.(n+1)(n+2)} =$ sum required.

14. Required the sum (s) of the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}$ &c. continued *ad infinitum*.

$$\text{Let } x = \frac{1}{2} \text{ and } s = \frac{z}{1+x}.$$

$$\text{Then } \frac{z}{1+x} = x(1 - x + x^2 - x^3 + x^4 \text{ \&c.})$$

$$\text{And } z = (1+x) \times (x - x^2 + x^3 - x^4 + x^5 \text{ \&c.})$$

Whence, by multiplication,

$$\begin{array}{l}
x - x^2 + x^3 - x^4 + x^5 \;\&c. \\
1 + x \\
\hline
x - x^2 + x^3 - x^4 + x^5 \;\&c. \\
\quad + x^2 - x^3 + x^4 - x^5 \;\&c. \\
\hline
\end{array}$$

Whose sum is $= x + 0 + 0 + 0 + 0$ &c.

Therefore $z = x$ and $x - x^2 + x^3 - x^4 + x^5 \;\&c. = \frac{x}{1+x}$.

Or $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32}$ &c. $= \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3} =$ sum required.

15. Required the sum of the series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} \;\&c.$$

Let $x = \frac{1}{2}$ and $s = \frac{z}{(1-x)^2}$.

Then $\frac{z}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + 5x^5$ &c.

And $z = (1-x)^2 \times (x + 2x^2 + 3x^3 + 4x^4 + 5x^5 \;\&c.)$

Whence, by multiplication,

$$\begin{array}{l}
x + 2x^2 + 3x^3 + 4x^4 \;\&c. \\
1 - 2x + x^2 \\
\hline
x + 2x^2 + 3x^3 + 4x^4 \;\&c. \\
\quad - 2x^2 - 4x^3 - 6x^3 \;\&c. \\
\qquad\qquad + x^3 + 2x^4 \;\&c. \\
\hline
\end{array}$$

Whose sum is $= x + 0 + 0 + 0$ &c.

Therefore $z = x$,

And $x + 2x^2 + 3x^3 + 4x^4 + 5x^5 \;\&c. = \frac{x}{(1-x)^2}$.

Or $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32}$ &c. $= \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2 =$ sum required.

16. Required the sum (s) of the infinite series

$$\frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \frac{25}{243} \;\&c.$$

Let $x=\frac{1}{3}$ and $\frac{z}{(1-x)^3}=s$.

Then $\frac{z}{(1-x)^3}=x+4x^2+9x^3+16x^4+25x^5$ &c.

And $z=(1-x)^3\times(x+4x^2+9x^3+16x^4\ \text{&c.})=x+x^2$, as will be found by actual multiplication.

Therefore $x+x^2=z$,

And $x+4x^2+9x^3+16x^4$ &c. $=\frac{x(1+x)}{(1-x)^3}$.

Or

$\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}$&c. $=\frac{\frac{1}{3}(1+\frac{1}{3})}{(1-\frac{1}{3})^3}=\frac{3}{2}=1\frac{1}{2}=$ sum required.

17. Required the sum (s) of the infinite series $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\frac{a+3d}{mr^3}$&c.

Let $x=\frac{1}{r}$, and $s=\frac{z}{m(1-x)^2}$.

Then $\frac{z}{m(1-x)^2}=\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\frac{a+3d}{mr^3}$&c.

Or $\frac{z}{(1-x)^2}=a+\frac{a+d}{r}+\frac{a+2d}{r^2}+\frac{a+3d}{r^3}$&c.

That is, $\frac{z}{(1-x)^2}=$

$a+(a+d)x+(a+2d)x^2+(a+3d)x^3+(a+4d)x^4$&c.

And $z=(1-x)^2\times\{a+(a+d)x+(a+2d)x^2+(a+3d)x^3\ \text{&c.}\}=(1-x)a+dx$, as will appear by actual multiplication.

Therefore $z=(1-x)a+dx$; and consequently $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}$&c. $=\frac{(1-r)a+dr}{m(1-r)^2}=$ sum of the infinite series required.

EXAMPLES FOR PRACTICE.

1. Required the sum of 100 terms of the series 2, 5, 8, 11, 14, &c. Ans. 15050

2. Required the sum of 50 terms of the series $1 + 2^2 + 3^2 + 4^2 + 5^2$ &c. Ans. 42925

3. Required the sum of the infinite series $1 + 3x + 6x^2 + 10x^3 + 15x^4$ &c. when x is less than 1.

Ans. $\frac{1}{(1-x)^3}$

4. Required the sum of the infinite series $1 + 4x^2 + 10x^2 + 20x^3 + 35x^4$ &c. when x is less than 1.

Ans. $\frac{1}{(1-x)^4}$

5. Required the sum of the infinite series $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9}$ &c. Ans. $\frac{5}{10}$, or $\frac{1}{2}$

6. Required the sum of 40 terms of the series $(1 \times 2) + (3 \times 4) + (5 \times 6) + (7 \times 8)$ &c.

Ans. 22960

7. Required the sum of n terms of the series $\frac{2x-1}{2x} + \frac{2x-3}{2x} + \frac{2x-5}{2x} + \frac{2x-7}{2x}$ &c. Ans. $n(\frac{2x-n}{2x})$

8. Required the sum of the infinite series $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6}$ &c. Ans. $\frac{1}{18}$

9. Required the sum of n terms of the series $\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \frac{5}{r^5} + \frac{6}{r^6}$ &c. Ans. $\frac{1}{(r-1)^2} - \frac{1}{r^n}\{\frac{nr+r-1}{(r-1)^2}\}$

10. Required the sum of n terms of the series $1 + 8x + 27x^2 + 64x^3 + 125x^4$ &c.

Ans. $\frac{1+4x+x^2}{(1-x)^4}$

11. Required the sum of n terms of the series $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \frac{1}{4.7}$ &c. . . . $\frac{1}{n(3+n)}$.

Ans. $\frac{n}{3+3n} + \frac{n}{12+6n} + \frac{n}{27+9n}$

12. Required the sum of the series $\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}$&c. . . . $\frac{1}{2n(4+2n)}$.

Ans. $\Sigma=\frac{3}{16}$, $s=\frac{5n+3n^2}{32+48n+16n^2}$

13. Required the sum of the series $\frac{1}{4.8}-\frac{1}{6.10}+\frac{1}{8.12}-$ &c. . . . $\frac{1}{(2+2n)(6+2n)}$.

Ans. $\Sigma=\frac{1}{48}$, $s=\frac{n}{16+16n}-\frac{n}{36+24n}$

14. Required the sum of the series $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\frac{1}{12.20}$&c. . . . $\frac{1}{3n(4+4n)}$.

Ans. $\Sigma=\frac{1}{12}$, $s=\frac{n}{12+12n}$

15. Required the sum of the series $\frac{6}{2.7}+\frac{6}{7.12}+\frac{6}{12.17}+\frac{6}{17.22}$&c. . . . $\frac{6}{(5n-3).(5n+2)}$.

Ans. $\Sigma=\frac{3}{5}$, $s=\frac{3n}{2+5n}$

16. Required the sum of the series $\frac{1}{8.18}+\frac{1}{10.21}+\frac{1}{12.24}+\frac{1}{14.27}$&c. . . . $\frac{1}{(6+2n).(15+3n)}$.

Ans. $\Sigma=\frac{3}{80}$, $s=\frac{n}{24(8+2n)}+\frac{n}{30(10+2n)}$

17. Required the sum of the series $\frac{1}{3.6}-\frac{1}{6.8}+\frac{1}{9.10}-\frac{1}{12.12}$&c. . . . $\frac{1}{3n(4+2n)}$.

Ans. $\Sigma=\frac{1}{24}$, $s=\frac{n}{2(2+6n)}-\frac{n}{4(6+6n)}$

18. Required the sum of n terms of the series $\frac{1}{1\ 2.4}+\frac{1}{2.3.5}+\frac{1}{3.4.6}+\frac{1}{4.5.7}$&c. . . . $\frac{1}{n(1+n).(3+n)}$.

Ans. $\frac{n}{3(1+n)}-\frac{n}{12(2+n)}-\frac{n}{18(3+n)}$

19. Required the sum of the series $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}$&c. . . . $\frac{1+n}{(1+2n).(3+2n)}$.

Ans. $\Sigma=\frac{1}{12}$, $s=\frac{1}{12}-\frac{1}{4(3+4n)}$

20. Required the sum of the series $\frac{1}{3}(\frac{2}{1.3})+\frac{1}{9}(\frac{3}{3.5})+\frac{1}{27}(\frac{4}{5.7})+\frac{1}{81}(\frac{5}{7.9})$&c. . . . $\frac{1+n}{3^n(2n-1).(2n+1)}$.

Ans. $\Sigma=\frac{1}{4}$, $s=\frac{1}{4}-\frac{1}{4.3^n(1+2n)}$

21. Required the sum of the series $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\frac{8}{4.5.6}$&c. . . . $\frac{4+n}{n(1+n).(2+n)}$.

Ans. $\Sigma=\frac{3}{2}$, $s=\frac{3}{2}-\frac{2}{1+n}+\frac{1}{2+n}$(g)

OF LOGARITHMS.

(M) LOGARITHMS are a set of numbers, which have been computed and formed into tables, for the purpose of facilitating many difficult arithmetical calculations; being so contrived, that the addition and subtraction of them answers to the multiplication and division of the natural numbers with which they are made to correspond (h).

(g) The series here treated of are such as are usually called algebraical; which, of course, embrace only a small part of the whole doctrine. Those, therefore, who may wish for farther information on this abstruse but highly curious subject, are referred to the Miscell. Analyt. of Demoivre, Sterling's Method. Differ., James Bernoulli de Seri. Infin., Simpson's Math. Dissert., Waring's Medit. Analyt., Clarke's translation of Lorgna's Series, the various works of Euler, and Lacroix Traité du Calcul Diff. et Int., where they will find nearly all the materials that have been hitherto collected respecting this branch of analysis.

Or, when taken in a similar but more general sense, logarithms may be considered as the exponents of the powers to which a given, or invariable, number must be raised, in order to produce all the common, or natural, numbers. Thus, if

$$a^x=y,\ a^{x'}=y',\ a^{x''}=y'',\ \&c.$$

then will the indices x, x', x'', &c. of the several powers of a, be the logarithms of the numbers y, y', y'', &c. in the scale, or system, of which a is the base.

So that, from either of these formulæ, it appears, that the logarithm of any number, taken separately, is the index of that power of some other number, which, when involved in the usual way, is equal to the given number.

And since the base a, in the above expressions, can be assumed of any value, greater or less than 1, it is plain that there may be an endless variety

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(h) This mode of computation, which is one of the happiest and most useful discoveries of modern times, is due to lord Napier, baron of Marchiston, in Scotland, who first published a table of these numbers, in the year 1614, under the title of *Canon Mirificum Logarithmorum;* which performance was eagerly received by the learned throughout Europe, whose efforts were immediately directed to the improvement and extension of the new calculus, that had so unexpectedly presented itself.

Mr. Henry Briggs, in particular, who was, at that time, professor of geometry in Gresham College, greatly contributed to of systems of logarithms, answering to the same natural numbers.

It is, likewise, farther evident, from the first of these equations, that when $y=1$, x will be $=0$, whatever may be the value of a; and consequently the logarithm of 1 is always 0, in every system of logarithms.

And if $x=1$, it is manifest, from the same equation, that the base a will be $=y$; which base is, therefore, the number whose proper logarithm, in the system to which it belongs, is 1.

Also, because $a^x=y$, and $a^{x'}=y'$, it follows, from the multiplication of powers, that $a^x \times a^{x'}$, or $a^{x+x'}=yy'$; and consequently, by the definition of logarithms, given above, $x+x'=\log.yy'$, or

$$\log.y+\log.y'=\log.yy'.$$

And, for similar reasons, if any number of the equations $a^x=y$, $a^{x'}=y'$, $a^{x''}=y''$, &c. be multiplied together, we shall have $a^{x+x'+x''\,\&c.}=yy'y''$ &c.; and therefore $x+x'+x''$ &c. $=\log.yy'y''$ &c.; or

$$\log.y+\log.y'+\log.y'' \,\&c. =\log.yy'y'' \,\&c.$$

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the advancement of this doctrine, not only by the very advantageous alteration, which he first introduced into the system of these numbers, by making 1 the logarithm of 10, instead of 2.3025852, as had been done by Napier; but also by the publication, in 1624 and 1633, of his two great works, the Arithmetica Logarithmica, and the Trigonometria Britanica, both of which were formed upon the principle above mentioned; as are, likewise, all our common logarithmic tables, at present in use.

See, for farther details on this part of the subject, the Introduction to my Treatise of Plane and Spherical Trigonometry, 8vo. 1806; and for the construction and use of the tables, consult those of Sherwin, Hutton, Taylor, Callet, and Borda, where every necessary information, of this kind, may be readily obtained.

From which it is evident, that the logarithm of the product of any number of factors is equal to the sum of the logarithms of those factors.

Hence, if all the factors of a given number, in any case of this kind, be supposed equal to each other, and the sum of them be denoted by m, the preceding property will then become

$$m \log. y = \log. y^m.$$

From which it appears, that the logarithm of the mth power of any number is equal to m times the logarithm of that number.

In like manner, if the equation $a^x = y$ be divided by $a^{x'} = y'$, we shall have, from the nature of powers, as before, $\frac{a^x}{a^{x'}}$, or $a^{x-x'} = \frac{y}{y'}$; and by the definition of logarithms, $x - x' = \log. \frac{y}{y'}$, or

$$\log. y - \log. y' = \log. \frac{y}{y'}.$$

Hence the logarithm of a fraction, or of the quotient arising from dividing one number by another, is equal to the logarithm of the numerator *minus* the logarithm of the denominator.

And if each member of the equation $a^x = y$ be raised to the fractional power $\frac{m}{n}$, we shall have $a^{\frac{m}{n}x} = y^{\frac{m}{n}}$; and consequently, as before, $\frac{m}{n}x = \log. y^{\frac{m}{n}}$;

$$\text{Or } \frac{m}{n} \log. y = \log. y^{\frac{m}{n}}.$$

Where it appears, that the logarithm of a mixed root, or power, of any number, is found by multiplying the logarithm of the given number by the numerator of the index of that power, and dividing the result by the denominator.

And if the numerator m, of the fractional index, be, in this case, taken equal to 1, the above formula will then become

$$\frac{1}{n}\log. y=\log. y^{\frac{1}{n}}.$$

From which it follows, that the logarithm of the nth root of any number is equal to the nth part of the logarithm of that number.

Hence, besides the use of logarithms in abridging the operations of multiplication and division, they are equally applicable to the raising of powers and extracting of roots; which are performed by simply multiplying the given logarithm by the index of the power, or dividing it by the number denoting the root,

But, although the properties here mentioned are common to every system of logarithms, it was necessary, for practical purposes, to select some one of them from the rest, and to adapt the logarithms of all the natural numbers to that particular scale.

And as 10 is the base of our present system of arithmetic, the same number has accordingly been chosen for the base of the logarithmic system, now generally used.

So that, according to this scale, which is that of the common logarithmic tables, the numbers

$$\ldots 10^{-4},\ 10^{-3},\ 10^{-2},\ 10^{-1},\ 10^{0},\ 10^{1},\ 10^{2},\ 10^{3},\ 10^{4},\ \&c.$$

Or

$$\ldots \frac{1}{10000},\ \frac{1}{1000},\ \frac{1}{100},\ \frac{1}{10},\ 1,\ 10,\ 100,\ 1000,\ 10000,\ \&c.$$

have for their logarithms

$$\ldots -4,\quad -3,\quad -2,\quad -1,\quad 0,\quad 1,\quad 2,\quad 3,\quad 4,\quad \&c.$$

Which are evidently a set of numbers in arith metical progression, answering to another set in geometrical progression; as is the case in every system of logarithms.

And therefore, since the common, or tabular, logarithm of any number (n) is the index of that power of 10, which is equal to the given number, it is plain, from the following equation,

$$10^x = n, \text{ or } 10^{-x} = \frac{1}{n},$$

that the logarithms of all the intermediate numbers, in the above series, may be assigned by approximation, and made to occupy their proper places in the general scale (*i*).

It is also evident, that the logarithms of 1, 10, 100, 1000, &c. being 0, 1, 2, 3, &c. respectively, the logarithm of any number, falling between 0 and 1, will be 0 and some decimal parts; that of a number between 10 and 100, 1 and some decimal parts; of a number between 100 and 1000, 2 and some decimal parts; and so on.

(*i*) From the latter of these formulæ $a^{-x} = \frac{1}{n}$, which is equivalent to $\frac{1}{a^x} = \frac{1}{n}$, it appears, that the greater n is, or the less the fraction $\frac{1}{n}$, the greater will be the index $-x$; and since, by augmenting n, $\frac{1}{n}$ may be made to approach continually towards 0, as its limit, it follows, that when the base a is greater than 1, the logarithm of 0 will be infinitely negative. It is also now generally allowed, by the most celebrated analysts, that the logarithms of negative quantities are imaginary; though this subject was the occasion of a very long discussion between Leibnitz and John Bernoulli, the latter of whom maintained, that they were the same as for real quantities.—See Traité du Calcul Deferentiel et Intégral, by Lacroix, page 314, vol. i, and page 166, vol. ii, where the proofs given by Euler, of their being imaginary, are fully stated.

And for a like reason, the logarithms of $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. or of their equals .1, .01, .001, &c. in the descending part of the scale, being -1, -2, -3, &c. the logarithm of any number, falling between 0 and .1, will be -1 and some positive decimal parts; that of a number between .1 and .01, -2 and some positive decimal parts; of a number between .01 and .001, -3 and some positive decimal parts; &c.

Hence, likewise, as the multiplying or dividing of any number by 10, 100, 1000, &c. is performed by barely increasing or diminishing the integral part of its logarithm by 1, 2, 3, &c. it is obvious, that all numbers, which consist of the same figures, whether they be integral, fractional, or mixed, will have, for the decimal part of their logarithms, the same positive quantity.

So that, in this system, the integral part of any logarithm, which is usually called its index, or characteristic, is always less by 1 than the number of integers which the natural number consists of; and for decimals, it is the number which denotes the distance of the first significant figure from the place of units.

Thus, according to the logarithmic tables in common use, we have

Numbers.	*Logarithms.*
1.36820	0.1361496
20.0500	1.3021144
335.260	2.5253817
.46521	$\bar{1}.6676490$
.06154	$\bar{2}.7891575$
&c.	&c.

Where the sign − is put over the index, instead of before it, when that part of the logarithim is negative, in order to distinguish it from the decimal part, which is always to be considered as +, or affirmative.

Also, agreeably to what has been before observed, the logarithm of 38540 being 4.5859117, the logarithms of any other numbers, consisting of the same figures, will be as follows:

Numbers.	*Logarithms.*
3854	3.5859117
385.4	2.5859117
38.54	1.5859117
3.854	0.5859117
.3854	$\bar{1}.5859117$
.03854	$\bar{2}.5859117$
.003854	$\bar{3}.5859117$

Which logarithms, in this case, differ only in their indices, the decimal, or positive part, being the same in them all (*k*).

(*k*) The great advantages attending the common, or Briggean system of logarithms, above all others, arise chiefly from the readiness with which we can always find the characteristic or integral part of any logarithm from the bare inspection of the natural number to which it belongs, and the circumstance, that multiplying or dividing any number by 10, 100, 1000, &c. only influences the characteristic of its logarithm, without affecting the decimal part. Thus, for instance, if i be made to denote the index, or integral part of the logarithm of any number N, and d its decimal part, we shall have log. $N = i + d$; log. $10^m \times N = (i + m) + d$; log. $\frac{N}{10^m} = (i - m) + d$; where it is plain that the decimal part of the logarithm, in each of these cases, remains the same.

And as the indices, or integral parts, of the logarithms of any numbers whatever, in this system, can always be thus readily found, from the simple consideration of the rule above mentioned, they are generally omitted in the tables, being left to be supplied by the operator, as occasion requires.

It may here, also, be farther added, that, when the logarithm of a given number, in any particular system, is known, it will be easy to find the logarithm of the same number in any other system, by means of the following equations,

$$a^x = n,\ \text{and}\ e^{x'} = n;\ \text{or}\ x = \log. n,\ \text{and}\ x' = \text{l}. n.$$

In which case, log. denotes the logarithm of n, in the system of which a is the base, and l. its logarithm in the system of which e is the base.

For, since $a^x = e^{x'}$, or $a^{\frac{x}{x'}} = e$; and $e^{\frac{x'}{x}} = a$, we shall have, for the base a, $\frac{x}{x'} = \log.\ e$, or $x = x'\log.\ e$; and for the base e, $\frac{x'}{x} = \text{l}. a$, or $x' = x\,\text{l}. a$; and consequently, by substitution, from the former equations,

$$\log. n = \text{l}. n \times \log. e;\ \text{or}\ \log. n = \text{l}. n \times \frac{1}{\text{l}. a}.$$

Where the multiplier $\log. e$, or its equal $\frac{1}{\text{l}. a}$, expresses the constant relation which the logarithms of n have to each other in the systems to which they belong.

But the only system of these numbers, deserving of notice, except that above described, is the one that furnishes what have been usually called hyperbolic or Neperian logarithms, the base e of which is 2.718281828459. . . .

Hence, in comparing these with the common or tabular logarithms, we shall have, by putting a in the latter of the above formulæ $= 10$, the expression

$$\log. n = \text{l}. n \times \frac{1}{\text{l}. 10}, \text{ or } \text{l}. n = \log. n \times \text{l}. 10.$$

Where log. in this case, denotes the common logarithm of the number n, and l. its hyperbolic logarithm; the constant factor $\frac{1}{\text{l}. 10}$, which is $\frac{1}{2.3025850929}$, or its equal .4342944819, being what is usually called the modulus of the common or tabular system of logarithms (*l*).

PROBLEM I.

To compute the logarithm of any of the natural numbers 1, 2, 3, 4, 5, &c.

RULE I.

1. Take the geometrical series, 1, 10, 100, 1000, 10000, &c. and apply to it the arithmetical series, 0, 1, 2, 3, 4, &c. as logarithms.

(*l*) It may here be remarked, that, although the common logarithms have superseded the use of hyperbolic or Neperian logarithms, in all the ordinary operations to which these numbers are generally applied, yet the latter are not without some advantages peculiar to themselves; being of frequent occurrence in the application of the Fluxionary Calculus, to many analytical and physical problems, where they are required for the finding of certain fluents, which could not be so readily determined without their assistance; on which account, great pains have been taken to calculate tables of hyperbolic logarithms, to a considerable extent, chiefly for this purpose.

Vega, in his Trigonometrical Tables, printed at Leipsic, in 1797, has given the hyperbolic logarithms of all numbers from 1 to 2000; and Mr. Barlow, of the Royal Military Academy, in a Collection of Mathematical Tables, now in the press, has given them for the first 10000 numbers.

2. Find a geometric mean between 1 and 10, 10 and 100, or any other two adjacent terms of the series, betwixt which the number proposed lies.

3. Between the mean, thus found, and the nearest extreme, find another geometrical mean, in the same manner; and so on, till you are arrived within the proposed limit of the number whose logarithm is sought.

4. Find, also, as many arithmetical means between the corresponding terms of the other series, 0, 1, 2, 3, 4, &c. in the same order as you found the geometrical ones, and the last of these will be the logarithm answering to the number required.

EXAMPLES.

1. Let it be required to find the logarithm of 9.

Here the proposed number lies between 1 and 10.

First, then, the log. of 10 is 1, and the log. of 1 is 0.

Therefore $\frac{1}{2}(1+0)=\frac{1}{2}=.5$ is the arithmetical mean,

And $\sqrt{(10\times1)}=\sqrt{10}=3.1622777$ is the geometrical mean;

Hence the log. of 3.1622777 is .5.

Secondly, the log. of 10 is 1, and the log. of 3.1622777 is .5.

Therefore $\frac{1}{2}(1+.5)=.75$ is the arithmetical mean,

And $\sqrt{(10\times3.1622777)}=5.6234132$ is the geometrical mean;

Hence the log. of 5.6234132 is .75.

Thirdly, the log. of 10 is 1, and the log. of 5.6234132 is .75;

Therefore $\frac{1}{2}(1+.75)=.875$ is the arithmetical mean,

And $\sqrt{(10\times 5.6234132)}=7.4989422$ is the geometrical mean;

Hence the log. of 7.4989422 is .875.

Fourthly, the log. of 10 is 1, and the log. of 7.4989422 is .875;

Therefore $\frac{1}{2}(1+.875)=.9375$ is the arithmetical mean,

And $\sqrt{(10\times 7.4989422)}=8.6596431$ is the geometrical mean;

Hence the log. of 8.6596431 is .9375.

Fifthly, the log. of 10 is 1, and the log. of 8.6596431 is .9375;

Therefore $\frac{1}{2}(1+.9375)=.96875$ is the arithmetical mean,

And $\sqrt{(10\times 8.6596431)}=9.3057204$ is the geometrical mean;

Hence the log. of 9.3057204 is .96875.

Sixthly, the log. of 8.6596431 is .9375, and the log. of 9.3057204 is .96875;

Therefore $\frac{1}{2}(.9375+.96875)=.953125$ is the arithmetical mean,

And $\sqrt{(8.6596431\times 9.3057204)}=8.9768713$ is the geometrical mean:

Hence the log. of 8.9768713 is .953125.

And, by proceeding in this manner, it will be found, after 25 extractions, that the logarithm of 8.9999998 is .9542425; which may be taken for the logarithm of 9, as it differs from it so little, that it may be considered as sufficiently exact for all practical purposes.

And in this manner were the logarithms of almost all the prime numbers at first computed.

RULE II.

When the logarithm of any number (n) is known, the logarithm of the next greater number may be readily found from the following series, by calculating a sufficient number of its terms, and then adding the given logarithm to their sum.

$$\text{Log.}(n+1)=\log.n+m\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\frac{1}{7(2n+1)^7}+\frac{1}{9(2n+1)^9}+\frac{1}{11(2n+1)^{11}}\&c.\right\}$$

Or

$$\text{Log.}(n+1)=\log.n+\left\{\frac{m}{2n+1}+\frac{\text{A}}{3(2n+1)^2}+\frac{3\text{B}}{5(2n+1)^2}+\frac{5\text{C}}{7(2n+1)^2}+\frac{7\text{D}}{9(2n+1)^2}+\frac{9\text{E}}{11(2n+1)^2}\&c.\right\}$$

Where A, B, C, &c. represent the terms immediately preceding those in which they are first used, and m = twice the modulus = .8685889638 (m).

(m) The difference between the logarithms of any two consecutive numbers, is so much the less as the numbers are greater.

For, taking the two following logarithmic equations,

$$a^x=n,\ a^{x+\delta}=n+1,$$

and dividing the second by the first, we shall have

$$a^\delta=1+\frac{1}{n},\ \text{or}\ \delta=\log.(1+\frac{1}{n})$$

where it is plain, that the greater the number n is, the less will be the fraction $\frac{1}{n}$; and, consequently, the less will δ (which is the

EXAMPLES.

1. Let it be required to find the common logarithm of the number 2.

Here, because $n+1=2$, and consequently $n=1$ and $2n+1=3$, we shall have

$$\frac{m}{2n+1}=\frac{.868588964}{3}\ldots=.289529654(\text{A})$$

$$\frac{\text{A}}{3(2n+1)^2}=\frac{.289529654}{3.3^2}\ldots=.010723321(\text{B})$$

$$\frac{3\text{B}}{5(2n+1)^2}=\frac{3\times.010723321}{5.3^2}=.000714888(\text{C})$$

$$\frac{5\text{C}}{7(2n+1)^2}=\frac{5\times.000714888}{7.3^2}=.000056737(\text{D})$$

$$\frac{7\text{D}}{9(2n+1)^2}=\frac{7\times.000056737}{9.3^2}=.000004903(\text{E})$$

$$\frac{9\text{E}}{11(2n+1)^2}=\frac{9\times.000004903}{11.3^2}=.000000446(\text{F})$$

$$\frac{11\text{F}}{13(2n+1)^2}=\frac{11\times.000000446}{13.3^2}=.000000042(\text{G})$$

$$\frac{\text{G}}{15(2n+1)^2}=\frac{13\times.000000042}{15.3^2}=.000000004(\text{H})$$

Sum of 8 terms . . .	.301029995
Add log. of 1	.000000000
Log. of 2	.301029995

Which logarithm is true to the last figure inclusively.

2. Let it be required to compute the logarithm of the number 3.

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difference of log. $(n+1)$ and log. n) or its equal log. $(1+\frac{1}{n})$, differ from the logarithm of 1, which is 0.

Here, since $n+1=3$, and consequently $n=2$, and $2n+1=5$, we shall have

$$\frac{m}{2n+1}=\frac{.868588964}{5}\ldots=.173717793(\text{A})$$

$$\frac{\text{A}}{3(2n+1)^2}=\frac{.173717793}{3.5^2}\ldots=.002316237(\text{B})$$

$$\frac{3\text{B}}{5(2n+1)^2}=\frac{3\times .002316237}{5.5^2}=.000055590(\text{C})$$

$$\frac{5\text{C}}{7(2n+1)^2}=\frac{5\times .000055590}{7.5^2}=.000001588(\text{D})$$

$$\frac{7\text{D}}{9(2n+1)^2}=\frac{7\times .000001588}{9.5^2}=.000000050(\text{E})$$

$$\frac{9\text{E}}{11(2n+1)^2}=\frac{9\times .000000050}{11.5^2}=.000000002(\text{F})$$

Sum of 6 terms 176091260
Log. of 2 301029995

Log. of 3 477121255

Which logarithm is also correct to the nearest unit in the last figure.

Hence, because the sum of the logarithms of any two numbers gives the logarithm of their product, and the difference of the logarithms gives the logarithm of the quotient; we may compute, from the above two logarithms, and the logarithm of 10, which is 1, a great number of other logarithms, as in the following examples:

3. Because 2 × 2 = 4, therefore log. 2 .301029995
mult. by 2 2

gives log. 4 .602059990

4. Because $2 \times 3 = 6$, therefore to log. 2	.301029995
add log. 3	.477121255
sum is log. 6	.778151250
5. Because $2^3 = 8$, therefore log. 2	.301029995
mult. by 3	3
gives log. 8	.903089985
6. Because $3^2 = 9$, therefore log. 3	477121255
mult. by 2	2
gives log. 9	.954242510
7. Because $\frac{10}{2} = 5$, therefore from log. 10	1.000000000
take log. 2	.301029995
leaves log. 5	.698970005
8. Because $3 \times 4 = 12$, therefore to log. 3	.477121255
add log. 4	.602059991
gives log. 12	1.079181246

And thus, by computing, according to this general rule, the logarithms of the next succeeding prime numbers 7, 11, 13, 17, 19, 23, &c. we can find, by composition and division, as many other logarithms as we please, or may speedily examine any logarithm in the table.

MULTIPLICATION BY LOGARITHMS.

Take out the logarithms of the factors from the table, and add them together; then the natural number answering to the sum will be the product required.

Observing, in the addition, that what is to be carried from the decimal part of the logarithms is always affirmative, and must, therefore, be added to the indices, or integral parts, after the manner of positive and negative quantities in algebra.

EXAMPLES.

1. Multiply 37.153 by 4.086, by logarithms.

Nos.	*Logs.*
37.153	1.5699939
4.086	0.6112984
Prod. 151.8071 .	2.1812923

2. Multiply 112.246 by 13.958, by logarithms.

Nos.	*Logs.*
112.246	2.0491709
13.958	1.1448232
Prod. 1566.729 .	3.1939941

3. Multiply 46.7512 by .3275, by logarithms.

Nos.	*Logs.*
46.7512	1.6697928
.3275	$\bar{1}.5152113$
Prod. 15.31102 .	1.1850041

Here, the + 1, that is to be carried from the decimals cancels the − 1, and consequently there remains 1 in the upper line to be set down.

4. Multiply .37816 by .04782, by logarithms.

Nos.	*Logs.*
.37816	$\bar{1}.5776756$
.04782	$\bar{2}.6796096$
Prod. .0180836	$\bar{2}.2572852$

Here the + 1 to be carried destroys the − 1, in the upper line, as before, and there remains the − 2 to be set down.

5. Multiply 3.768, 2.053, and .007693, together.

Nos.	*Logs.*
3.768	0.5761109
2.053	0.3123889
.007693	$\bar{3}.8860997$
Prod. .059511	$\bar{2}.7745995$

Here the + 1 to be carried from the decimals, when added to − 3, makes − 2, to be set down.

6. Multiply 3.586, 2.1046, .8372, and .0294, together.

Nos.	*Logs.*
3.586	0.554610
2.1046	0.323170
.8372	$\bar{1}.922829$
.0294	$\bar{2}.468347$
Prod. .1857618	$\bar{1}.268956$

Here the + 2 to be carried, cancels the − 2, and there remains the − 1 to be set down.

7. Multiply 4.0763 by 9.8432, by logarithms.
Ans. 40.12383

8. Multiply 498.256 by 41.2467, by logarithms.
Ans. 20551.41

9. Multiply 4.026747 by .012345, by logarithms.
Ans. .0497102

10. Multiply 3.12567, .02868, and .12379, together, by logarithms. Ans. .01109705

11. Multiply 2876.9, .10674, .098762, and .0031598, by logarithms. Ans. .0958299

DIVISION BY LOGARITHMS.

From the logarithm of the dividend, as found in the tables, subtract the logarithm of the divisor, and the natural number, answering to the remainder, will be the quotient required.

Observing, if the subtraction cannot be made in the usual way, to add, as in the former rule, the 1 that is to be carried from the decimal part, when it occurs, to the index of the logarithm of the divisor, and then this result, with its sign changed, to the remaining index, for the index of the logarithm of the quotient.

EXAMPLES.

1. Divide 4768.2 by 36.954, by logarithms.

Nos.	*Logs.*
4768.2	3.6783545
36.954	1.5676615
Quot. 129.032 .	2.1106930

2. Divide 21.754 by 2.4678, by logarithms.

Nos.	*Logs.*
21.754	1.3375391
2.4678	0.3923100
Quot. 81518 . . .	0.9452291

3. Divide 4.6257 by .17608, by logarithms.

Nos.	*Logs.*
4.6257	0.6651725
.17608	$\bar{1}$.2457100
Quot. 2.62741 .	1.4194675

Here −1, in the lower index, is changed into +1, whch is then taken for the index of the result.

4. Divide .27684 by 5.1576, by logarithms.

Nos.	*Logs.*
.27684	$\bar{1}$.4422288
5.1576	0.7124477
Quot. .0536761 .	$\bar{2}$.7297811

Here the 1 that is to be carried is taken as −1, and then added to −1, in the upper index, which gives −2 for the index of the result.

5. Divide 6.9875 by .075789, by logarithms.

Nos.	*Logs.*
6.9875	0.8443218
.075789	$\bar{2}$.8796062
Quot. 92.1967 . .	1.9647156

Here the 1, that is to be carried, is added to −2, which makes -1, and this put down with its sign changed is $+1$.

6. Divide .19876 by .0012345, by logarithms.

Nos.	*Logs.*
.19876	$\bar{1}.2983290$
.0012345 . . .	$\bar{3}.0914911$
Quot. 161.0051 .	2.2068379

Here -3, in the lower index, is changed into $+3$, and this added to -1, the other index, gives $+3-1$ or 2.

7. Divide 125 by 1728, by logarithms.
Ans. .0723379

8. Divide 1728.95 by 1.10678, by logarithms.
Ans. 1562.144

9. Divide .1023674 by 4.96523, by logarithms.
Ans. 2.061685

10. Divide 19956.7 by .048235, by logarithms.
Ans. 413739

11. Divide .067859 by 1234.59, by logarithms.
Ans. .0000549648

THE RULE OF THREE, OR PROPORTION, BY LOGARITHMS.

For any single proportion, add the logarithms of the second and third terms together, and subtract the logarithm of the first from their sum, according to the foregoing rules; then the natural number answering to the result will be the fourth term required.

But if the proportion be compound, add together the logarithms of all the terms that are to be multiplied, and from the result take the sum of the logarithms of the other terms, and the remainder will be the logarithm of the term sought.

Or, the same may be performed more conveniently thus,

Find the complement of the logarithm of the first term of the proportion, or what it wants of 10, by beginning at the left hand, and taking each of its figures from 9, except the last significant figure, on the right, which must be taken from 10; then add this result and the logarithms of the other two terms together, and the sum, abating 10 in the index, will be the logarithm of the fourth term, as before.

Or, if two or more logarithms are to be subtracted, as in the latter part of the above rule, add their complements and the logarithms of the terms to be multiplied together, and the result, abating as many 10's in the index as there are logarithms to be subtracted, will be the logarithm of the term required: observing, when the index of the logarithm, whose complement is to be taken, is negative, to add it, as if it were affirmative, to 9; and then take the rest of the figures from 9, as before.

EXAMPLES.

1. Find a fourth proportional to 37.125, 14.768, and 135.279, by logarithms.

Log. of 37.125	1.5696665
Complement	8.4303335
Log. of 14.768	1.1693217
Log. of 135.279 . . .	2.1312304
Ans. 53.81099 . .	1.7308856

2. Find a fourth proportional to .05764, .7186, and .34721, by logarithms.

Log. of .05764 . . .	$\bar{2}.7607240$
Complement	11.2392760
Log. of .7186 . . .	$\bar{1}.8564872$
Log. of .34721 . . .	$\bar{1}.5405922$
Ans. 4.328681 . .	0.6363554

3. Find a third proportional to 12.796 and 3.24718, by logarithms.

Log. of 12.796 . . .	1.1070742
Complement . . .	8.8929258
Log. 3.24718 . . .	0.5115064
Log. 3.24718 . . .	0.5115064
Ans. .8240216 . .	.9159386

4. Find the interest of 279*l.* 5*s.* for 274 days, at 4½ per cent. per annum, by logarithms.

Comp. log. 100 . . .	8.0000000
Comp. log. 365 . . .	7.4377071
Log. of 279.25 . . .	2.4459932
Log. of 274	2.4377506
Log. of 4.5	0.6532125
Ans. 9.433296*l.* . .	0.9746634

5. Find a fourth proportional to 12.678, 14.065, and 100.979, by logarithms. Ans. 112.0263

6. Find a fourth proportional to 1.9864, .4678, and 50.4567, by logarithms. Ans. 11.88262

7. Find a fourth proportional to .09658, .24958, and .008967, by logarithms. Ans. .02317234

8. Find a mean proportional between .498621 and 2.9587, by logarithms. Ans. 17.55623

INVOLUTION, OR THE RAISING OF POWERS, BY LOGARITHMS.

Take out the logarithm of the given number from the tables, and multiply it by the index of the proposed power; then the natural number, answering to the result, will be the power required.

Observing, if the index of the logarithm be negative, that this part of the product will be negative; but as what is to be carried from the decimal part will be affirmative, the index of the result must be taken accordingly (*n*).

EXAMPLES.

1. Find the square of 2.7568, by logarithms.

Log. of 2.7568 . . .	0.4402477
	2
Square 7.599946 . .	0.8804954

2. Find the cube of 7.0851, by logarithms.

Log. 7.0851 . . .	0.8503399
	3
Cube 355.6475 . . .	2.5510197

(*n*) In some examples of this rule, the resulting logarithm, when the question is wrought in the usual way, may come out wholly negative, in which case, as it cannot be found in that form in the tables, the decimal part of it must be subtracted from 1, and then −1, or $\bar{1}$, be put before the remainder, as

3. Find the fifth power of .087451, by logs.

$$\begin{array}{lr}
\text{Log. of .87451 . . .} & \bar{1}.9417648 \\
 & 5 \\
\hline
\text{Fifth power .5114695 .} & \bar{1}.7088240 \\
\hline
\end{array}$$

Where 5 times the negative index $\bar{1}$, being -5, and $+4$ to carry, the index of the power is $\bar{1}$.

4. Find the 365th power of 1.0045, by logs.

$$\begin{array}{lr}
\text{Log. 1.0045 . . .} & 0.0019499 \\
 & 365 \\
\hline
 & 97495 \\
 & 116994 \\
 & 58497 \\
\hline
\text{Power 5.148888 .} & 0.7117135 \\
\hline
\end{array}$$

---

a negative index; which will evidently adapt it to the purpose required, without altering its value.

Thus, if it were required to find the value of $(.07)^{.07}$ by logarithms, the operation, agreeably to the above rule, would stand thus:

$$\begin{array}{lr}
\text{Log. .07 . . .} & -2+.845098 \\
 & .07 \\
\hline
 & +.05915686 \\
 & -.14 \\
\hline
 & -.08084314 \\
 & +1. \\
\hline
 & \bar{1}.91915686 \\
\end{array}$$

where the natural number answering to the result is .8301539; which is the value required.

Or the same answer may be obtained more concisely, in a different way, by taking .14 from .05915686.

5. Required the square of 6.05987, by logarithms.
Ans. 36.72203

6. Required the cube of .176546, by logarithms.
Ans. .005502674

7. Required the 4th power of .076543, by logarithms. Ans. .0000343259

8. Required the 7th power of 1.09684, by logarithms. Ans. 1.909864

EVOLUTION, OR THE EXTRACTION OF ROOTS, BY LOGARITHMS.

Take out the logarithm of the given number from the table, and divide it by 2 for the square root, 3 for the cube root, &c. and the natural number answering to the result will be the root required.

But if it be a compound root, or one that consists both of a root and a power, multiply the logarithm of the given number by the numerator of the index, and divide the product by the denominator, for the logarithm of the root sought.

Observing, in either case, when the index of the logarithm is negative, and cannot be divided without a remainder, to increase it by such a number as will render it exactly divisible; and then carry the units borrowed, as so many tens, to the first figure of the decimal part, and divide the whole accordingly.

EXAMPLES.

1. Find the square root of 27.465, by logs.

Log. of 27.465 . . . 2)1.4387796

Root 5.2407 7193898

2. Find the cube root of 35.6415, by logs.

Log. 35.6415 . . . 3)1.5519560

Root 3.29093 . . .5173186

3. Find the 5th root of 7.0825, by logarithms.

Log. 7.0825 . . . 5)0.8501866

Root 1.479235 . . .1700373

4. Find the 365th root of 1.045, by logarithms.

Log. 1.045 . . 365)0.0191163

Root 1.000121 . . 0.0000524

5. Find the value of $(.001234)^{\frac{2}{3}}$ by logs.

Log. 001234 . . . $\bar{3}$.0913152

2

3)$\bar{6}$.1826304

Ans. .00115047 . . $\bar{2}$.0608768

Here, the divisor 3 being contained exactly twice in the negative index -6, the index of the quotient, to be put down, will be -2.

Find the value of $(.024554)^{\frac{3}{2}}$, by logarithms.

Log. of .024554 . . $\bar{2}$.3901223

3

2)$\bar{5}$.1703669

Ans. .00384754 . . $\bar{3}$.5851834

Here 2 not being contained exactly in -5, 1 is added to it, which gives -3 for the quotient; and the 1 that is borrowed being carried to the next figure, makes 11, which, divided by 2, gives .58 &c.

7. Required the square root of 365.5674, by logarithms. Ans. 19.11981

8. Required the cube root of 2.987635, by logarithms. Ans. 1.440265

9. Required the 4th root of .967845, by logarithms. Ans. .9918624

10. Required the 7th root of .098674, by logarithms. Ans. .7183146

11. Required the value of $(\frac{21}{373})^{\frac{2}{3}}$, by logarithms. Ans. 1.146895

12. Required the value of $(\frac{112}{1727})^{\frac{3}{5}}$, by logarithms. Ans. .1937115

MISCELLANEOUS QUESTIONS IN LOGARITHMS.

1. Required the square root of $\frac{2}{123}$, by logs. Ans. .1275153

2. Required the cube root of $\frac{1}{3.14159}$, by logs. Ans. .6827842

3. Required the .07 power of .00563, by logs. Ans. .6958821

4. Required the value of $\frac{(\frac{2}{3})^{\frac{1}{2}} \times (\frac{3}{4})^{\frac{1}{3}}}{17\frac{1}{3}}$, by logs. Ans. .04279825

5. Required the value of $\frac{1}{7}\sqrt{\frac{5}{8}} \times .012\sqrt[3]{\frac{7}{11}}$, by logs. Ans. .001165713

6. Required the value of $\frac{\frac{1}{9}\sqrt{11\frac{3}{4}} \times .03\sqrt[3]{15\frac{1}{5}}}{7\frac{1}{3}\sqrt[3]{12\frac{1}{5}} \times .9\sqrt[4]{17\frac{1}{8}}}$, by logs. Ans. .0009158638

7. Required the value of $\frac{127}{4}(\frac{\frac{5}{6}\sqrt{19} + \frac{4}{7}\sqrt[3]{35\frac{1}{3}}}{14\frac{7}{10} - \frac{1}{11}\sqrt[5]{28\frac{2}{3}}})$, by logarithms. Ans. 49.38712

(P) A SYNOPSIS OF SOME OF THE MOST USEFUL FORMS OF SERIES,

In which a, b, c, &c. are the initial terms of the series, d the common difference, r the ratio, n any number of terms, s the sum of n terms, and Σ the sum of an infinite number of terms.

Series arising from Division.

$$x \pm \frac{x^2}{a} + \frac{x^3}{a^2} \pm \frac{x^4}{a^3} + \frac{x^5}{a^4} \pm \frac{x^6}{a^5} + \text{\&c.}$$

$$\Sigma = \frac{ax}{a \mp x}$$

$$x^{m-1} + ax^{m-2} + a^2x^{m-3} + \;.\;.\;.\; + a^{m-2}x + a^{m-1}$$

$$s = \frac{x^m - a^m}{x - a}$$

Where m is any whole or integral number.

Arithmetical Series.

$$a + (a+d) + (a+2d) + (a+3d) + (a+4d) + \;.\;.\;.\;.\;.\;.\;.\;.\; \{a + (n-1)d\}$$

$$s = \frac{n}{2}\{2a + (n-1)d\}$$

$$a + (a-d) + (a-2d) + (a-3d) + (a-4d) + \;.\;.\;.\;.\;.\;.\;.\;.\; \{a - (n-1)d\}$$

$$s = \frac{n}{2}\{2a - (n-1)d\}$$

Geometrical Series.

$$a + ra + r^2a + r^3a + r^4a + \;.\;.\;.\;.\; r^{n-1}a$$

$$s = \frac{r^n - 1}{r - 1}a$$

Or, if r be a proper fraction, in the continued series

$$a + ra + r^2a + r^3a + r^4a + r^5a + \text{\&c.}$$

$$\Sigma = \frac{a}{1 - r}$$

Series of the powers of arithmeticals.

$$a^2+(a+d)^2+(a+2d)^2+(a+3d)^2+\&c.$$

$$s=na^2+n(n-1)ad+\frac{n}{6}(n-1)(2n-1)d^2$$

$$a^3+(a+d)^3+(a+2d)^3+(a+3d)^3+\&c.$$

$$s=na^3+\frac{3n}{2}(n-1)a^2d+\frac{n}{2}(n-1)(2n-1)ad^2+$$

$$\frac{n^2}{4}(n-1)^2d^3$$

&c. &c.

Hence, taking a and d in the former series, and those above given, $=1, 2, 3$, &c. respectively, we shall have

$$1+2+3+4+\ldots\ldots+n=\frac{n^2+n}{2}$$

$$1^2+2^2+3^2+4^2+\ldots+n^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$$

$$1^3+2^3+3^3+4^3+\ldots+n^3=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$$

$$1^4+2^4+3^4+4^4+\ldots+n^4=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$$

$$1^5+2^5+3^5+4^5+\ldots+n^5=\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}$$

&c. &c.

Series arising from the expansion of a binomial.

$$a^m\{1+m(\frac{x}{a})+\frac{m(m-1)}{1.2}(\frac{x}{a})^2+\frac{m(m-1)(m-2)}{1.2.3}(\frac{x}{a})^3+\&c.\}$$

Or

$$a^m+ma^{m-1}x+\frac{m(m-1)}{1.2}a^{m-2}x^2+\frac{m(m-1)(m-2)}{1.2.3}a^{m-3}x^3\&c.$$

$$s=(a+x)^m$$

$$a^m\{1-m(\frac{x}{a})+\frac{m(m-1)}{1.2}(\frac{x}{a})^2-\frac{m(m-1)(m-2)}{1.2.3}(\frac{x}{a})^3+\&c.\}$$

Or

$$a^m-ma^{m-1}x+\frac{m(m-1)}{1.2}a^{m-2}x^2-\frac{m(m-1)(m-2)}{1.2.3}a^{m-3}x^3\&c.$$

$$s=(a-x)^m$$

Each of which series will terminate when m is a whole number.

Series derived from a binomial surd.

$$a^{\frac{m}{n}}\left\{1+\frac{m}{n}\left(\frac{x}{a}\right)+\frac{m}{n}\cdot\frac{m-n}{2n}\left(\frac{x}{a}\right)^2+\frac{m}{n}\cdot\frac{m-n}{2n}\cdot\frac{m-2n}{3n}\left(\frac{x}{a}\right)^3\&c.\right\}$$

Or

$$a^{\frac{m}{n}}\left\{1+\frac{m}{n}\left(\frac{x}{a}\right)+\frac{m-n}{2n}\mathrm{A}\left(\frac{x}{a}\right)+\frac{m-2n}{3n}\mathrm{B}\left(\frac{x}{a}\right)+\frac{m-3n}{4n}\mathrm{C}\left(\frac{x}{a}\right)\&c.\right\}$$

$$\Sigma=(a+x)^{\frac{m}{n}}$$

$$a^{\frac{m}{n}}\left\{1+\frac{m}{n}\left(\frac{x}{a+x}\right)+\frac{m}{n}\cdot\frac{m+n}{2n}\left(\frac{x}{a+x}\right)^2+\frac{m}{n}\cdot\frac{m+n}{2n}\cdot\frac{m+2n}{3n}\left(\frac{x}{a+x}\right)^3\&c.\right\}$$

Or

$$a^{\frac{m}{n}}\left\{1+\frac{m}{n}\left(\frac{x}{a+x}\right)+\frac{m+n}{2n}\mathrm{A}\left(\frac{x}{a+x}\right)+\frac{m+2n}{3n}\mathrm{B}\left(\frac{x}{a+x}\right)+\frac{m+3n}{4n}\mathrm{C}\left(\frac{x}{a+x}\right)\&c.\right\}$$

$$\Sigma=(a+x)^{\frac{m}{n}}$$

$$(2a)^{\frac{m}{n}}\left\{1-\frac{m}{n}\left(\frac{a-x}{a+x}\right)+\frac{m}{n}\cdot\frac{m+n}{2n}\left(\frac{a-x}{a+x}\right)^2-\frac{m}{n}\cdot\frac{m+n}{2n}\cdot\frac{m+2n}{3n}\left(\frac{a-x}{a+x}\right)^3\&c.\right\}$$

Or

$$(2a)^{\frac{m}{n}}\left\{1-\frac{m}{n}\left(\frac{a-x}{a+x}\right)+\frac{m+n}{2n}\mathrm{A}\left(\frac{a-x}{a+x}\right)-\frac{m+2n}{3n}\mathrm{B}\left(\frac{a-x}{a+x}\right)+\frac{m+3n}{4n}\mathrm{C}\left(\frac{a-x}{a+x}\right)\&c.\right\}$$

$$\Sigma=(a+x)^{\frac{m}{n}}$$

Reciprocals of the above series.

$$\frac{1}{a^{\frac{m}{n}}}\left\{1-\frac{m}{n}\left(\frac{x}{a}\right)+\frac{m}{n}\cdot\frac{m+n}{2n}\left(\frac{x}{a}\right)^2-\frac{m}{n}\cdot\frac{m+n}{2n}\cdot\frac{m+2n}{3n}\left(\frac{x}{a}\right)^3\&c.\right\}$$

Or

$$\frac{1}{a^{\frac{m}{n}}}\{1-\frac{m}{n}(\frac{x}{a})+\frac{m+n}{2n}A(\frac{x}{a})-\frac{m+2n}{3n}B(\frac{x}{a})+\frac{m+3n}{4n}C(\frac{x}{a})\ \&c.\}$$

$$\Sigma=\frac{1}{(a+x)^{\frac{m}{n}}}$$

$$\frac{1}{a^{\frac{m}{n}}}\{1-\frac{m}{n}(\frac{x}{a+x})+\frac{m}{n}\cdot\frac{m-n}{2n}(\frac{x}{a+x})^2-\frac{m}{n}\cdot\frac{m-n}{2n}\cdot\frac{m-2n}{3n}(\frac{x}{a+x})^3\&c.\}$$

Or

$$\frac{1}{a^{\frac{m}{n}}}\{1-\frac{m}{n}(\frac{x}{a+x})+\frac{m-n}{2n}A(\frac{x}{a+x})-\frac{m-2n}{3n}B(\frac{x}{a+x})+\frac{m-3n}{4n}C(\frac{x}{a+x})\&c.\}$$

$$\Sigma=\frac{1}{(a+\ )^{\frac{m}{n}}}$$

$$\frac{1}{(2a)^{\frac{m}{n}}}\{1+\frac{m}{n}(\frac{a-x}{a+x})+\frac{m}{n}\cdot\frac{m-n}{2n}(\frac{a-x}{a+x})^2+\frac{m}{n}\cdot\frac{m-n}{2n}\cdot\frac{m-2n}{3n}(\frac{a-x}{a+x})^3\&c.\}$$

Or

$$\frac{1}{(2a)^{\frac{m}{n}}}\{1+\frac{m}{n}(\frac{a-x}{a+x})+\frac{m-n}{2n}A(\frac{a-x}{a+x})+\frac{m-2n}{3n}B(\frac{a-x}{a+x})+\frac{m-3n}{4n}C(\frac{a-x}{a+x})\&c.\}$$

$$\Sigma=\frac{1}{(a+x)^{\frac{m}{n}}}$$

Sum and difference of two binomial surds.

$$2a^{\frac{m}{n}}\{1+\frac{m}{n}\cdot\frac{m-n}{2n}(\frac{x}{a})^2+\frac{m-2n}{3n}\cdot\frac{m-3n}{4n}A(\frac{x}{a})^2+\frac{m-4n}{5n}\cdot\frac{m-5n}{6n}B(\frac{x}{a})^2\&c.\}$$

$$\Sigma=(a+x)^{\frac{m}{n}}+(a-x)^{\frac{m}{n}}$$

$$2a^{\frac{m}{n}}\{\frac{m}{n}(\frac{x}{a}) + \frac{m-n}{2n}\cdot\frac{m-2n}{3n}\text{A}(\frac{x}{a})^2 + \frac{m-3n}{4n}\cdot$$
$$\frac{m-4n}{5n}\text{B}(\frac{x}{a})^2\&\text{c.}\}$$

$$\Sigma = (a+x)^{\frac{m}{n}} - (a-x)^{\frac{m}{n}}$$

In all of which cases it is to be observed, that A, B, C, &c. denote the terms immediately preceding those in which they are first found.

Series of figurate numbers.

$$1 + m +$$
$$\frac{m(m+1)}{1.2} + \frac{m(m+1)(m+2)}{1.2.3} + \frac{m(m+1)(m+2)(m+3)}{1.2.3.4}\&\text{c.}$$

$$\text{S} = \frac{n(n+1)(n+2)(n+3)\ \&\text{c. to } m \text{ terms}}{1.2.3.4\ \&\text{c. to } m \text{ terms}}$$

Hence by, putting $m = 1, 2, 3, 4$, &c. successively, we shall have

$$1 + 1 + 1 + 1 + 1 + \&\text{c.} = n$$

$$1 + 2 + 3 + 4 + 5 + \&\text{c.} = \frac{n(n+1)}{2}$$

$$1 + 3 + 6 + 10 + 15 + \&\text{c.} = \frac{n(n+1)(n+2)}{2.3}$$

$$1 + 4 + 10 + 20 + 35 + \&\text{c.} = \frac{n(n+1)(n+2)(n+3)}{2.3.4}$$

$$1 + 5 + 15 + 35 + 70 + \&\text{c.} = \frac{n(n+1)(n+2)(n+3)(n+4)}{2.3.4.5}$$

&c. &c.

Series of compound figurate numbers.

$$1 + mx +$$
$$\frac{m(m+1)}{2}x^2 + \frac{m(m+1)(m+2)}{2.3}x^3 + \frac{m(m+1)(m+2)(m+3)}{2.3.4}x^4\ \&\text{c.}$$

$$\Sigma = \frac{1}{(1-x)^m}$$

And, by putting $m = 1, 2, 3$, &c. successively, we shall have

$$1+x+x^2+x^3+x^4+x^5\text{\&c.}=\frac{1}{1-x}\quad\ldots\;(\Sigma)$$

$$1+2x+3x^2+4x^3+5x^4+6x^5+\text{ \&c.}=\frac{1}{(1-x)^2}$$

$$1+3x+6x^2+10x^3+15x^4+21x^5\text{\&c.}=\frac{1}{(1-x)^3}$$

$$1+4x+10x^2+20x^3+35x^4+56x^5\text{\&c.}=\frac{1}{(1-x)^4}$$

$$1+5x+15x^2+35x^3+70x^4+126x^5\text{\&c.}=\frac{1}{(1-x)^5}$$

$$1+6x+21x^2+56x^3+126x^4+252x^5\text{\&c.}=\frac{1}{(1-x)^6}$$

&c. &c.

Series of the reciprocals of figurate numbers.

$$1+\frac{1}{m}+$$

$$\frac{1.2}{m(m+1)}+\frac{1.2.3}{m(m+1)(m+2)}+\frac{1.2.3.4}{m(m+1)(m+2)(m+3)}\text{\&c.}$$

$$\Sigma=\frac{m-1}{m-2}$$

And by putting $m=1, 2, 3,$ &c. successively, we shall have

$$1+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}\text{\&c.}=\infty$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\text{\&c.}=\infty$$

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}\text{\&c.}=2$$

$$1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}\text{\&c.}=\frac{3}{2}$$

$$1+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}\text{\&c.}=\frac{4}{3}$$

$$1+\frac{1}{6}+\frac{1}{21}+\frac{1}{56}+\frac{1}{126}\text{\&c.}=\frac{5}{4}$$

&c. &c.

Series of compound geometrical numbers.

$$(\Sigma)$$

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + 5^2x^4 \text{ \&c.} = \frac{1+x}{(1-x)^3}$$

$$1 + 2^3x + 3^3x^2 + 4^3x^3 + 5^3x^4 \text{ \&c.} = \frac{1+4x+x^2}{(1-x)^4}$$

$$1 + 2^4x + 3^4x^2 + 4^4x^3 + 5^4x^4 \text{ \&c.} = \frac{1+11x+11x^2+x^3}{(1-x)^5}$$

$$1 + 2^5x + 3^5x^2 + 4^5x^3 + 5^5x^4 \text{ \&c.} = \frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6}$$

&c. &c.

Series of certain fractional numbers.

$$\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+3)(n+4)} \text{ \&c.}$$

$$\Sigma = \frac{1}{n}$$

$$\frac{1}{n(n+b)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} \text{ \&c.}$$

$$\Sigma = \frac{1}{2n(n+1)}$$

$$\frac{1}{n(n+1)(n+2)(n+3)} + \frac{1}{(n+1)(n+2)(n+3)(n+4)} + \text{\&c.}$$

$$\Sigma = \frac{1}{3n(n+1)(n+2)}$$

Where the law of continuation is sufficiently manifest.

Differential series.

$$a - bx + cx^2 - dx^3 + ex^4 - fx^5 + \text{\&c.}$$

$$\Sigma = a - \frac{bx}{1+x} - \frac{D'x^2}{(1+x)^2} - \frac{D''x^3}{(1+x)^3} - \frac{D'''x^4}{1+x^4} - \text{\&c.}$$

$$a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \text{\&c.}$$

$$\Sigma = a + \frac{bx}{1-x} + \frac{D'x^2}{(1-x^2)} + \frac{D''x^3}{(1-x^3)} + \frac{D'''x^4}{(1-x^4)} + \text{\&c.}$$

Where D′, D″, D‴, &c. are the first terms of the several orders of differences of *a*, *b*, *c*, *d*, &c.;

That is $\mathrm{D}' = b - c$, $\mathrm{D}'' = b - 2c + d$, $\mathrm{D}''' = b - 3c + 3d - e$, $\mathrm{D}^{\mathrm{iv}} = b - 4c + 6d - 4e + f$, $\mathrm{D}^{\mathrm{v}} = b - 5c + 10d - 10e + 5f - g$, &c.

Binomial series, the terms of which are respectively multiplied by α, β, γ, δ, &c.

$$\alpha a^m + \beta b a^{m-1} x + \gamma c a^{m-2} x^2 + \delta d a^{m-3} x^3 \,\&c.$$

$$\mathrm{s} = \alpha(a + x)^m + \mathrm{D}' bx(a + x)^{m-1} + \mathrm{D}'' cx^2(a + x)^{m-2} + \mathrm{D}''' x^3(a + x)^{m-3} \,\&c.$$

Where D', D'', D''', &c. denote the first terms of the several orders of differences of the quantities α, β, γ, &c. as before.

And if p, $p+q$, $p+2q$, $p+3q$, &c. be put for α, β, γ, &c. we shall have

$$pa^m + (p + q) ba^{m-1}x + (p + 2q) ca^{m-2}x^2 + (p+3q) da^{m-3}x^3 \,\&c.$$

$$\mathrm{s} = p(a + x)^m + qbx(a + x)^{m-1}$$

$$\frac{\alpha}{x} + \frac{\beta}{x^2} + \frac{\gamma}{x^3} + \frac{\delta}{x^4} + \frac{\varepsilon}{x^5} + \&c.$$

$$\Sigma = \frac{\alpha}{x-1} + \frac{\mathrm{D}'}{(x-1)^2} + \frac{\mathrm{D}''}{(x-1)^3} + \frac{\mathrm{D}'''}{(x-1)^4} \,\&c.$$

Binomial series, the terms of which are divided by factors in arithmetical progression.

$$\frac{a^m}{2.3\,.\,.\,r} + \frac{ba^{m-1}x}{2.3\,.\,.\,.\,(r+1)} + \frac{ca^{m-2}x^2}{2.3\,.\,.\,.\,(r+2)} + \frac{da^{m-3}x^3}{2.3\,.\,.\,.\,(r+3)} \,\&c.$$

$$\Sigma \text{ or } \mathrm{s} = \frac{(a + x)^{m+r}}{(m+1)(m+2)(m+3)\,.\,.\,.\,(m+r)x^r}.$$

Hence we have the following particular series:

$$a^m + \frac{ba^{m-1}x}{2} + \frac{ca^{m-2}x^2}{3} + \frac{da^{m-3}x^3}{4} \,\&c.$$

$$S = \frac{(a+x)^{m+1} - a^{m+1}}{(m+1)x}$$

$$\frac{a^m}{1.2} + \frac{ba^{m-1}x}{2.3} + \frac{ca^{m-2}x^2}{3.4} + \frac{da^{m-3}x^3}{4.5} \&c.$$

$$S = \frac{(a+x)^{m+2} - a^{m+2} - (m+2)a^{m+1}x}{(m+1)(m+2)x^2}$$

$$\frac{a^m}{1.2.3} + \frac{ba^{m-1}x}{2.3.4} + \frac{ca^{m-2}x^2}{3.4.5} + \frac{da^{m-3}x^3}{4.5.6} \&c.$$

$$S = \frac{(a+x)^{m+3} - a^{m+3} - (m+3)a^{m+2}x - \frac{1}{2}(m+3)(m+2)a^{m+1}x^2}{(m+1)(m+2)(m+3)x^3}$$

Where the law of continuation of any series of this kind is evident.

Binomial series, the terms of which are respectively divided by the terms of a series of figurate numbers.

$$a^m + \frac{ba^{m-1}x}{r+1} + \frac{ca^{m-2}x^2}{(r+1)\frac{1}{2}(r+2)} + \frac{da^{m-3}x^3}{(r+1)\frac{1}{2}(r+2)\frac{1}{3}(r+3)} + \&c.$$

$$\Sigma = \frac{(a+x)^{m+r}}{(m+1)\frac{1}{2}(m+2)\frac{1}{3}(m+3)\ .\ .\ .\ \frac{1}{r}(m+r)x^r}$$

From which may be derived a number of particular series, by expounding r, by 1, 2, 3, &c. respectively.

LOGARITHMIC SERIES.

$$\text{Log. } (1+n) = \frac{1}{M}(n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \frac{1}{5}n^5 - \&c.)$$

$$\text{Log. } (\frac{1}{1-n}) = \frac{1}{M}(n + \frac{1}{2}n^2 + \frac{1}{3}n^3 + \frac{1}{4}n^4 + \frac{1}{5}n^5 + \&c.)$$

$$\text{Log. } (\frac{1+n}{1-n}) = \frac{2}{M}(n + \frac{1}{3}n^3 + \frac{1}{5}n^5 + \frac{1}{7}n^7 + \frac{1}{9}n^9 + \&c.)$$

$$\text{Log. } (\frac{n+1}{n}) = \frac{2}{M}(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \frac{1}{7(2n+1)^7} \&c.)$$

From which formulæ we can readily obtain the following logarithms:

$$\text{Log. } n = \frac{1}{\text{M}}\left\{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \frac{1}{4}(n-1)^4 + \&c.\right\}$$

$$\text{Log. } n = \frac{1}{\text{M}}\left\{\frac{n-1}{n} + \frac{1}{2}\left(\frac{n-1}{n}\right)^2 + \frac{1}{3}\left(\frac{n-1}{n}\right)^3 + \frac{1}{4}\left(\frac{n-1}{n}\right)^4 + \&c.\right\}$$

$$\text{Log. } n = \frac{2}{\text{M}}\left\{\frac{n-1}{n+1} + \frac{1}{3}\left(\frac{n-1}{n+1}\right)^3 + \frac{1}{5}\left(\frac{n-1}{n+1}\right)^5 + \frac{1}{7}\left(\frac{n-1}{n+1}\right)^7 + \&c.\right\}$$

Or the same forms may be expressed more generally thus:

$$\text{Log. } (n+z) = \log. n + \frac{1}{\text{M}}\left\{\frac{z}{n} - \frac{1}{2}\left(\frac{z}{n}\right)^2 + \frac{1}{3}\left(\frac{z}{n}\right)^3 - \frac{1}{4}\left(\frac{z}{n}\right)^4 + \&c.\right\}$$

$$\text{Log. } (n-z) = \log. n - \frac{1}{\text{M}}\left\{\frac{z}{n} + \frac{1}{2}\left(\frac{z}{n}\right)^2 + \frac{1}{3}\left(\frac{z}{n}\right)^3 + \frac{1}{4}\left(\frac{z}{n}\right)^4 + \&c.\right\}$$

$$\text{Log. } (n+z) = \log. n + \frac{2}{\text{M}}\left\{\frac{z}{2n+z} + \frac{1}{3}\left(\frac{z}{2n+z}\right)^3 + \frac{1}{5}\left(\frac{z}{2n+z}\right)^5 + \&c.\right\}$$

$$\text{Log. } (n-z) = \log. n - \frac{2}{\text{M}}\left\{\frac{z}{2n-z} + \frac{1}{3}\left(\frac{z}{2n-z}\right)^3 + \frac{1}{5}\left(\frac{z}{2n-z}\right)^5 + \&c.\right\}$$

To which there may also be added the following:

$$\text{Log. } n = \frac{1}{\text{M}}\left\{(n-n^{-1}) - \frac{1}{2}(n-n^{-2})^2 + \frac{1}{3}(n-n^{-3})^3 - \frac{1}{4}(n-n^{-4})^4 \&c.\right\}$$

$$\text{Log. } n = \frac{r}{\text{M}}\left\{(\sqrt[r]{n}-1) - \frac{1}{2}(\sqrt[r]{n}-1)^2 + \frac{1}{3}(\sqrt[r]{n}-1)^3 - \frac{1}{4}(\sqrt[r]{n}-1)^4 \&c.\right\}$$

$$\text{Log. } n = \frac{r}{\text{M}}\left\{\left(1-\frac{1}{\sqrt[r]{n}}\right) + \frac{1}{2}\left(1-\frac{1}{\sqrt[r]{n}}\right)^2 + \frac{1}{3}\left(1-\frac{1}{\sqrt[r]{n}}\right)^3 + \frac{1}{4}\left(1-\frac{1}{\sqrt[r]{n}}\right)^4 \&c.\right\}$$

Where r denotes any root whatever of n; and consequently may be so taken that the value of $\sqrt[r]{n}$, found by repeated extractions of the square root, shall differ but little from 1.

The following series may also be used for determining the number answering to a given logarithm.

$$n = 1 + \frac{\log. n}{M} + \frac{1}{2}\left(\frac{\log. n}{M}\right)^2 + \frac{1}{2.3}\left(\frac{\log. n}{M}\right)^3 + \frac{1}{2.3.4}\left(\frac{\log. n}{M}\right)^4 \&c.$$

In all of which cases M must be taken $=1$, for hyperbolic logarithms, and $=2.302585093$, for common logarithms.

Exponential series.

$$n^x =$$

$$1 + \frac{x}{1}\left(\frac{\log. n}{M}\right) + \frac{x^2}{2}\left(\frac{\log. n}{M}\right)^2 + \frac{x^3}{2.3}\left(\frac{\log. n}{M}\right)^3 + \frac{x^4}{2.3.4}\left(\frac{\log. n}{M}\right)^4 \&c.$$

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \&c.$$

Where $e=$ number where hyperbolic or Neperian logarithm is 1, which is known to be 2.7182818284.

MISCELLANEOUS QUESTIONS.

1. A person being asked what o'clock it was, answered, that it was between eight and nine, and that the hour and minute hands were exactly together; what was the time?

Ans. 8h. 43min. $38\frac{2}{11}$sec.

2. A certain number, consisting of two places of figures, is equal to the difference of the squares of its digits, and if 36 be added to it the digits will be inverted; what is the number? Ans. 48

3. It is required to find three numbers such, that

$\frac{3}{8}$ of the first, $\frac{5}{14}$ of the second, and $\frac{7}{20}$ of the third, shall be all equal to each other.

Ans. 280, 294, and 300

4. What two numbers are those, whose difference, sum, and product, are to each other as the numbers 2, 3, and 5, respectively? Ans. 2 and 10

5. A person, in a party at cards, betted three shillings to two upon every deal, and after twenty deals found he had gained five shillings; how many deals did he win? Ans. 13

6. A person wishing to enclose a piece of ground with palisadoes, found, if he set them a foot asunder, that he should have too few by 150, but if he set them a yard asunder he should have too many by 70; how many had he? Ans. 180

7. A cistern will be filled by two cocks, A and B, running together, in twelve hours, and by the cock A alone in twenty hours; in what time will it be filled by the cock B alone? Ans. 30 hours

8. Given $(x+1) \times (x^2+1) \times (x^3+1) = 30x^3$, to find the value of x by a quadratic equation.

Ans. $x = \frac{1}{2}(3 \pm \sqrt{5})$

9. Required that arithmetical progression, whose number of terms is 10, sum of the terms 185, and sum of the cubes of the terms 104525.

Ans. 5, 8, 11, 14, 17, 20, 23, 26, 29, 32

10. A, B, and C, are to share 100000*l.* between them, in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, respectively; but C's part being lost, by death, it is required to divide the whole sum properly between the other two.

Ans. A's share $57142\frac{282}{329}$*l.* and B's $42857\frac{47}{329}$*l.*

11. If three agents, A, B, C, can produce the effects a, b, c, in the times e, f, g, respectively; in what time would they jointly produce the effect d.

Ans. $d \div \left(\frac{a}{e} + \frac{b}{f} + \frac{c}{g}\right)$

12. What number is that, which being severally added to 3, 19, and 51, shall make the results in geometrical progression? Ans. 13

13. It is required to find two mean proportionals between 3 and 24; and four mean proportionals between 3 and 96.

Ans. 6 and 12; and 6, 12, 24, and 48.

14. It is required to find six numbers in geometrical progression such, that their sum shall be 315, and the sum of the two extremes 165.

Ans. 5, 10, 20, 40, 80, and 160

15. The sum of two numbers is a, and the sum of their reciprocals is b; required the numbers.

Ans. $\frac{a}{2} \pm \frac{1}{2}\sqrt{\left\{\frac{a}{b}(ab-4)\right\}}$

16. The sum of three numbers in harmonical proportion is 13, and their continued product is 72; what are the numbers? Ans. 6, 4, and 3

17. A certain number of men were employed in making a fish pond, and when they had been at work 24 days, and had half finished it, 16 men more were set on, by which the remaining half was completed in 16 days: how many men were employed at first; and what was the whole expence, at 1*s.* 6*d.* a day per man?

Ans. 32 men; and the whole expence 115*l.* 4*s.*

18. It is required to find two numbers such, that if the square of the first be added to the second, the

sum shall be 62, and if the square of the second be added to the first, it shall be 176.

Ans. 7 and 13

19. The fore wheel of a carriage makes six revolutions more than the hind wheel, in going 120 yards; but if the circumference of each wheel was increased by three feet, it would make only four revolutions more than the hind wheel in the same space; what is the circumference of each wheel?

Ans. 12 and 15 feet

20. A number consisting of three digits in arithmetical progression being divided by the sum of its digits gives 26, and if 198 be added to it the digits will be inverted; what is the number?

Ans. 234

21. If 6000 soldiers be placed in the form of an oblong, so that the number of men in rank shall be to those in file as 3 to 2, how much ground will they stand on, supposing the distance between man and man to be $2\frac{3}{4}$ yards?

Ans. 92ac. 3ro. 35p. $7\frac{4961}{7744}$yards

22. It is required to divide a given number a into two such parts, x and y, that the sum of mx and ny shall be equal to some other given number b.

Ans. $x=\frac{b-an}{m-n}$ and $y=\frac{am-b}{m-n}$

23. Out of a pipe of wine, containing 84 gallons, 10 gallons were drawn off, and the vessel replenished with 10 gallons of water; after which, 10 gallons of the mixture were again drawn off, and then 10 gallons more of water poured in: now the like process having been repeated four times, it is required to find how much pure wine remained in

the vessel, supposing the two fluids to have been thoroughly mixed each time? Ans. $48\frac{4}{5}$ gallons

24. It is required to find the least whole number such, that the sum of its aliquot parts shall exceed itself by 7. Ans. 196

25. It is required to find two such numbers, x and y, that the sum of the aliquot parts of x shall be equal to $\frac{2}{3}y$, and the sum of the aliquot parts of y, equal to $\frac{3}{4}x$. Ans. $x=20$, $y=33$

26. A sum of money is to be divided equally among a certain number of persons; now if there had been 3 claimants less, each would have had 150*l.* more, but if there had been 6 more, each would have had 120*l.* less; required the number of persons, and the sum divided.

Ans. 9 persons, sum 2700*l.*

27. From each of sixteen pieces of gold, a person filed the worth of half a crown, and then offered them in payment for their original value, but the fraud being detected, and the pieces weighed, they were found to be worth, in the whole, no more than eight guineas; what was the original value of each piece? Ans. 13*s.*

28. A composition of tin and copper, containing 100 cubic inches, was found to weigh 505 ounces; how many ounces of each did it contain, supposing the weight of a cubic inch of copper to be $5\frac{1}{4}$ ounces, and that of a cubic inch of tin $4\frac{1}{4}$ ounces.

Ans. 420 oz. of copper, and 85 oz. of tin

29. A privateer, running at the rate of 10 miles

an hour, discovers a vessel 18 miles a head of her, making way at the rate of 8 miles an hour; how many miles will the latter run before she is overtaken. Ans. 72 miles

30. At what time, next after 12 o'clock, will the hour, minute, and second hand of a watch, be all at the greatest distance from each other?

Ans. 12 hours, $21\frac{9}{11}$ minutes

31. The sum of three numbers in harmonical proportion is 39, and the sum of their squares 549; what are the numbers? Ans. 9, 12, and 18

32. Given the sum of two numbers $=2$, and the sum of their ninth powers $=32$, to find the numbers by a quadratic equation.

Ans. $1\pm\sqrt{(2\sqrt{34}-11)}$

33. It is required to find two numbers such, that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

Ans. $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5+\sqrt{5})$

34. The arithmetical mean of two numbers exceeds the geometrical mean by 13, and the geometrical mean exceeds the harmonical mean by 12; what are the numbers? Ans. 234 and 104

35. Given $x^3y+y^3x=3$, and $x^6y^2+y^6x^2=7$, to find the values of x and y.

Ans. $x=\frac{1}{2}(\sqrt{5}+1)$, $y=\frac{1}{2}(\sqrt{5}-1)$

36. Given $x^3+xy=1932$ and $y^3-xy=4709$ to find the values of x and y. Ans. $x=12$ $y=17$

37. Given $x^3y+y^3x=512500$ and $x^2y-y^2x=5500$ to find the values of x and y.

Ans $x=25$ and $y=20$

38. Given $x+y+z=23$, $xy+xz+yz=167$, and $xyz=385$, to find x, y, and z.

Ans. $x=5$, $y=7$, $z=11$

39. To find four numbers, x, y, z, and w, having the product of every three of them given; *viz.* $xyz=231$, $xyw=420$, $yzw=1540$, and $xzw=660$.

Ans. $x=3$, $y=7$, $z=11$, and $w=20$

40. Given $x+yz=384$, $y+xz=237$, and $z+xy=192$, to find the values of x, y, and z.

Ans. $x=10$, $y=17$, and $z=22$

41. Given $x^2+xy=108$, $y^2+yz=69$, and $z^2+xz=580$, to find the values of x, y, and z.

Ans. $x=9$, $y=3$, and $z=20$

42. Given $x^2+xy+y^2=5$ and $x^4+x^2y^2+y^4=11$ to find the values of x and y by a quadratic.

Ans. $x=\frac{2}{5}\sqrt{10}+\frac{1}{5}\sqrt{5}$, $y=\frac{2}{5}\sqrt{10}-\frac{1}{5}\sqrt{5}$

43. The equation $x^{4n}-2x^{3n}+x^n=a$ being given, it is required to find the value of x.

Ans. $x=\frac{1}{2}+\sqrt[n]{\{\frac{3}{4}\pm\sqrt{(a+\frac{3}{4})}\}}$

44. It is required to find the number by which a people must increase annually, so that they may be doubled at the end of every century.

Ans. By 144, *nearly*

45. Supposing a certain number of persons to increase every year by a $\frac{1}{10}$ part, how long will it be before there are 10 times as many as there were at first?

Ans. 231 years *nearly*

46. Supposing there are 100000 inhabitants in

a province, and that its population increases every year by a $\frac{1}{30}$ part; what will be the number of its inhabitants at the end of a century?

Ans. 2654874

47. The earth having been repeopled, after the deluge, by the three sons of Noah and their wives, it is required to find what number of inhabitants there would be at the time of the building of the tower of Babel, 210 years afterwards, supposing their numbers to have been doubled every 14 years?

Ans. 196608

48. Required the least number of weights, and the weight of each, that will weigh any number of pounds from one pound to twenty-nine hundred weight.

Ans. 1, 3, 9, 27, 81, 243, 729, and 2187

49. It is required to find two square numbers such, that either of them, when added to its aliquot parts, shall make the same sum.

Ans. 106276 and 165649

50. It is required to find four whole numbers such, that the square of the greatest may be equal to the sum of the squares of the other three.

Ans. 3, 4, 12, and 13

51. It is required to find the least number, which being divided by 6, 5, 4, 3, and 2, shall leave the remainders 5, 4, 3, 2, and 1, respectively.

Ans. 59

52. Given the cycle of the sun 18, the golden number 8, and the Roman indiction 10, to find the year.

Ans. 1717

53. It is required to find the least integral values of x and y in the equation $x^2 - 13y^2 = 4$.

Ans. $x = 119$ and $y = 33$

54. Given $5x + 7y + 9z = 93256$, to find all the different solutions in affirmative integers, which the equation admits of. Ans. 13801148

55. It is required to find two different isosceles triangles such, that their perimeters and areas shall be both expressed by the same numbers.

Ans. Sides of the one 29, 29, 40; and of the other 37, 37, 24

56. It is required to find the sides of three right angled triangles, in whole numbers, such, that their areas shall be all equal to each other.

Ans. 58, 40, 42; 74, 24, 70; 113, 15, 112

57. Given $x^{\frac{1}{x}} = 1.2655$, to find a near approximate value of x. Ans. 3.82013

58. Given $x^y = 5000$, and $y^x = 3000$, to find the values of x and y.

Ans. $x = 4.691445$, and $y = 5.510132$

59. Given $y^x = 9\frac{1}{2}$, and $x^x + y^y = 32\frac{1}{4}$, to find the values of x and y.

Ans. $x = 2.038809$, and $y = 3.016867$

60. Given $x^x - x = y$, and $y^x + y = x$, to find the values of x and y.

Ans. $x = 1.747933$, and $y = .906164$

61. Given $x^x + y^y = 285$, and $y^x - x^y = 14$, to find the values of x and y.

Ans. $x = 4.016698$, and $y = 2.825716$

62. To find a square number such, that the sum of all its aliquot parts shall be a square.

Ans. 2401

63. To find two whole numbers such, that if unity be added to each of them, and also to their halves, the sums, in both cases, shall be squares.

Ans. 48 and 1680

64. Required the two least nonquadrate numbers x and y such, that x^2+y^2 and x^3+y^3 shall be both squares. Ans. $x=364$ and $y=273$

65. It is required to find two whole numbers such, that their sum shall be a cube, and their product and quotient squares.

Ans. 25 and 100, or 100 and 900, &c.

66. To find two square numbers such, that their sum shall be a square and their difference a cube, and the side of the said square and cube equal to each other. Ans. $\frac{782}{15625}$ and $\frac{441}{15625}$

67. To find three whole numbers such, that if to the square of each the product of the other two be added, the sums shall be all squares.

Ans. 9, 73, and 328

68. To find three numbers such, that if each of them be added to the cube of their sum, the sums, thus arising, shall be all cubes.

Ans. $\frac{23625}{157464}$, $\frac{1538}{157464}$, and $\frac{18577}{157464}$

69. To find three numbers such, that if each of them be subtracted from the cube of their sum, the three remainders shall be all cubes.

Ans. $\frac{13851}{85184}$, $\frac{19467}{85184}$, and $\frac{18954}{85184}$

70. It is required to find three biquadrate numbers such, that their sum shall be a square.

Ans. 12^4, 15^4, and 20^4

71. It is required to find three numbers in con-

tinued geometrical progression such, that their three differences shall be all squares.

Ans. 567, 1008, and 1792

72. It is required to find three whole numbers such, that the sum or difference of any two of them shall be square numbers.

Ans. 434657, 420968, and 150568

73. It is required to find two whole numbers such, that their sum shall be a square, and the sum of their squares a biquadrate.

Ans. 4565486027761 and 1061652293520

74. It is required to find three whole square numbers such, that the difference between the sum of every two of them and the third shall be square numbers. Ans. $(149)^2$, $(241)^2$, and $(269)^2$

75. It is required to find four whole numbers such, that the difference of every two of them shall be a square number.

Ans. 1873432, 2288168, 2399057, and 6560657

76. It is required to find the logarithm of the number $\frac{1}{3}\sqrt{3}$, in the system of which 9 is the base.

Ans. $-.25$, or $-\frac{1}{4}$

77. It is required to find how many figures it would take to express the 25th term of the series $2^1 + 2^2 + 2^4 + 2^8 + 2^{16}$ &c. Ans. 5050446 figures

78. Required the sum of the series $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} +$ &c. continued to infinity. Ans. $\frac{3}{4}$

79. Required the sum of the infinite series $\frac{3}{4} - \frac{9}{16} + \frac{27}{64} - \frac{81}{256} +$ &c. Ans. $\frac{3}{7}$

80. Required the sum of the series $5 + 6 + 7 + 8 + 9 +$ &c. continued to n terms. Ans. $\frac{n}{2}(n+9)$

81. Given $1+3x+6x^2+10x^3+15x^4+$ &c. in infinitum $=10$, to find the value of x.

Ans. $x=1-\frac{1}{10}\sqrt[3]{100}$

82. Given $\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\frac{1}{3840}x^5+$ &c. in infinitum $=\frac{1}{4}$, to find the value of x.

Ans. $x=\frac{1}{2}-\frac{1}{16}+\frac{1}{96}-\frac{1}{512}+\frac{1}{2560}-$ &c.

83. It is required to find the sum of 100 terms of the series $(1\times2)+(3\times4)+(5\times6)+(7\times8)+(9\times10)$&c. Ans. 343400

84. Required the sum of $1^2+2^2+3^2+4^2+5^2$&c. . . . $+50^2$, which gives the number of shot in a square pile, the side of which is 50. Ans. 42925

85. Required the sum of 25 terms of the series $35+36\times2+37\times3+38\times4+39\times5$ &c. which gives the number of shot in a complete oblong pile, consisting of 25 tiers, the number of shot in the uppermost row being 35. Ans. 16575

86. Required the approximate value of the series $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}$&c. continued ad infinitum.

Ans. .822467

87. It is required to find the approximate value of the hypergeometrical series $1-1+2-6+24-120+720$ &c. continued ad infinitum.

Ans. .5963473

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